95. On Some Euler Products. I

By Nobushige KUROKAWA
Department of Mathematics, Tokyo Institute of Technology

(Communicated by Kunihiko KODAIRA, M. J. A., Nov. 12, 1984)

§1. **Prime sets.** We say that a set P is a "prime set" if P is a countable infinite set having a real valued "norm function" $N: P\to \boldsymbol{R}$ satisfying the following: (1) $N(p)>1$ for all $p\in P$, and (2) $N(p_i)\to\infty$ as $i\to\infty$ for an (i.e., any) ordering $P=\{p_1, p_2, \cdots\}$. Put $\pi(t, P)=\sharp\{p\in P; N(p)\leqq t\}$ for $t>0$ where $\sharp$ denotes the cardinality. Then, (2) is equivalent to that $\pi(t, P)$ is finite for each $t>0$. We define $d(P)=\inf\{d>0; \sum_p N(p)^{-d}<\infty\}$. Then $0\leqq d(P)\leqq\infty$, and we have

$$d(P)=\limsup_{t\to\infty}\frac{\log\pi(t, P)}{\log t}.$$

We are exclusively interested in the case of finite $d(P)$, and we define the zeta function of P by $\zeta(s, P)=\prod_p (1-N(p)^{-s})^{-1}$ for a variable s in the complex numbers $\boldsymbol{C}$. This infinite product (an Euler product over P) converges absolutely in $\mathrm{Re}(s)>d(P)$. When $0<d(P)<\infty$, by defining another norm function by $N^1(p)=N(p)^{d(P)}$, we can normalize (P, N) to (P, N^1) which satisfies $d(P)=1$.

Example 1. Let A be a commutative finitely generated $\boldsymbol{Z}$-algebra, where $\boldsymbol{Z}$ denotes the ring of rational integers. Let $M(A)$ be the category of A-modules, and let $P=P(M(A))=P(A)$ be the "set" of all isomorphism classes of simple objects of $M(A)$. In this case P is actually a set and is consisting of isomorphism classes of simple A-modules. For each $p\in P$, let $N(p)=\sharp p$ be the cardinality of p as a set. (Each p is a finite set.) Then P is a prime set with the (integer valued) norm function N, and $d(P)$ is equal to the Krull dimension $\dim(A)$ of A. In particular, when $A=\boldsymbol{Z}$, $\zeta(s, P(\boldsymbol{Ab}))$ is equal to the Riemann zeta function $\zeta(s)$, where $\boldsymbol{Ab}=M(\boldsymbol{Z})$ is the category of abelian groups. (Note that $P(\boldsymbol{Ab})$ is the set of isomorphism classes of simple abelian groups, and that a simple abelian group is a finite cyclic group of prime order.) In other words, the Riemann zeta function is the zeta function of the category $\boldsymbol{Ab}$. In general, we expect that:

$$Z(s, P)=\zeta(s, P)\Gamma(s, P)=\prod_{m=0}^{2d(P)} Z_m(s, P)^{(-1)^{m+1}}$$

with the gamma factor $\Gamma(s, P)$, where $Z_m(s, P)$ is holomorphic on $\boldsymbol{C}$ having the functional equation for $s\to m-s$ with all zeros on $\mathrm{Re}(s)=m/2$. When $\zeta(s, P)$ is meromorphic on $\boldsymbol{C}$, we have an "explicit formula" attached to $\zeta(s, P)$ in the form $\sum_p M(p)=\sum_\lambda W(\lambda)$, where λ

runs over zeros and poles of $\zeta(s, P)$. Hence, for $\boldsymbol{P}=(P_1, \cdots P_r)$ with $P_i=P(A_i)$, we have a "multiple explicit formula" $\sum_{\boldsymbol{p}} M(\boldsymbol{p})=\sum_{\boldsymbol{\lambda}} W(\boldsymbol{\lambda})$, where $\boldsymbol{p}=(p_1, \cdots, p_r)$ and $\boldsymbol{\lambda}=(\lambda_1, \cdots, \lambda_r)$. Specializing this formula we have a "multiple zeta function" of order being equal to the sum of orders of $\zeta(s, P_i)$, which has a "multiple Euler product" expression. This zeta function is considered to correspond to a multiple category. (We remark that in this example the commutativity of A is not essential, and we have an analogous example in the non-commutative case also.)

Example 2. Let C be a category with a zero (or "null") object 0. We assume that C is a concrete category in the sense that C is a subcategory of ***Set*** (the category of sets). We say that a non-zero object X of C is simple if each morphism (or "arrow") $f: X\to Y$ is zero or monic for any object Y. (If C is abelian, this condition is equivalent to that X has only two subobjects 0 and X.) We say that X is a finite simple object if X is a simple object with finite cardinality $N(X)=\sharp X$ as a set. (More generally, we may define a concrete category as a pair (C, F) of a category C and a faithful functor $F: C\to \boldsymbol{Set}$; then $N(X)=\sharp F(X)$.) We denote by $P=P(C)$ the class (hopefully a set) of all isomorphism classes of finite simple objects of C. The above Example 1 is the case of $C=\boldsymbol{M}(A)$, where the finiteness is satisfied automatically. As a non-abelian example, let $C=\boldsymbol{Grp}$ be the category of groups. Then $P=P(\boldsymbol{Grp})$ is the set of isomorphism classes of finite simple groups, which is a prime set with $d(P)=1$, and $\zeta(s, P(\boldsymbol{Grp}))$ is holomorphic in $\mathrm{Re}(s)>1/3$ except for a simple pole at $s=1$. (Here we use the classification of finite simple groups.) We remark that as a weaker candidate for "primes" we may take the class $P'(C)$ of all isomorphism classes of finite "indecomposable objects" instead of finite simple objects in some cases such as $\boldsymbol{M}(A)$ and $\boldsymbol{Grp}$, but the associated zeta functions are not so good in general; for example $\zeta(s, P'(\boldsymbol{Ab}))$ is meromorphic in $\mathrm{Re}(s)>0$ with the natural boundary $\mathrm{Re}(s)=0$. Moreover $d(P'(\boldsymbol{Grp}))=\infty$. We note that the category $\boldsymbol{Set}_*$ of pointed sets is also a non-abelian example, where we have $\zeta(s, P(\boldsymbol{Set}_*))=(1-2^{-s})^{-1}$ and $\zeta(s, P'(\boldsymbol{Set}_*))=\zeta(s)$.

Example 3. Let X be a scheme of finite type over $\mathrm{Spec}(\boldsymbol{Z})$. Let $P=P(X)$ be the set of all closed points of X. Then P is a prime set with the usual norm function, and we have $d(P)=\dim(X)$. If $X=\mathrm{Spec}(A)$ with A being as in Example 1, then $P(A)$ and $P(X)$ are identified (norm-preserving) since each $p\in P(A)$ is written as $p=A/\boldsymbol{m}$ for a maximal ideal $\boldsymbol{m}$ of A.

§2. Euler products. We introduce L-functions. Let P be a prime set with finite $d(P)$. Let G be a topological group, and $\mathrm{Conj}(G)$

be the set of all conjugacy classes of G. Let $\alpha: P \to \mathrm{Conj}(G)$ be a map. We call such a triple $E=(P, G, \alpha)$ an Euler datum. We define $\bar{E}=(P, G\times \boldsymbol{R}, \bar{\alpha})$ by $\bar{\alpha}(p)=(\alpha(p), \log N(p))$. We denote by $\mathrm{Irr}^u(G)$ the set of all equivalence classes of irreducible finite dimensional continuous unitary representations of G, and denote by $R^u(G)$ the ring of virtual characters generated (spanned) over $\boldsymbol{Z}$ by $\{\mathrm{tr}(\rho); \rho \in \mathrm{Irr}^u(G)\}$, where $\mathrm{tr}(\rho)$ denotes the trace of ρ. Let T be an indeterminate, and $H(T)$ be a polynomial belonging to $1+T\cdot R^u(G)[T]$. For each $c \in \mathrm{Conj}(G)$ we denote by $H_c(T)$ the polynomial belonging to $1+T\cdot \boldsymbol{C}[T]$ obtained from $H(T)$ by taking values at c of the coefficients. (Remark that elements of $R^u(G)$ are class functions on G.) We say that $H(T)$ is unitary if for each $c \in \mathrm{Conj}(G)$ there is a unitary matrix M_c such that $H_c(T)=\det(1-M_cT)$ or $H_c(T)=1$. We define

$$L(s, E, H)=\prod_p H_{\alpha(p)}(N(p)^{-s})^{-1},$$

which is (at least) meromorphic (not necessarily holomorphic) in $\mathrm{Re}(s) > d(P)$. We say that $E=(P, G, \alpha)$ is complete if the following hold: (1) if $H(T)$ is unitary then $L(s, E, H)$ is meromorphic on $\boldsymbol{C}$, and (2) if $H(T)$ is not unitary then $L(s, E, H)$ is meromorphic in $\mathrm{Re}(s)>0$ with the natural boundary $\mathrm{Re}(s)=0$. Note that if $\bar{E}$ is complete then E is also. We have a general condition making $\bar{E}$ (and E) complete, which is described by properties of $L(s, E, \rho)=L(s, E, D_\rho)$ for $\rho \in \mathrm{Irr}^u(G)$ where $D_\rho(T)=\det(1-\rho T)$. (In §3 we note an example.) We note the following point. Let P be a prime set, and let $N(P)=\{N_1, N_2, \cdots\}$ be the image of N, where $1<N_1<N_2<\cdots$. We denote by m_i the multiplicity of N_i defined by $m_i=\#\{p \in P; N(p)=N_i\}=\pi(N_i, P)-\pi(N_{i-1}, P)$. (We have $N_i\to\infty$ as $i\to\infty$, and $1\leqq m_i<\infty$.) We define:

$$\mu(P)=\limsup_{i\to\infty} \frac{\log m_i}{\log N_i}.$$

Then we have $0\leqq \mu(P)\leqq d(P)$. (This is shown directly; it follows also from $m_i\leqq \pi(N_i, P)$ and the previous expression for $d(P)$.) For our applications the case where $\mu(P)<d(P)$ is important.

Remark 1. An Euler datum $E=(P, G, \alpha)$ is especially important if G is a "universal (or, generic) fundamental group" in a suitable sense. Note that for Examples 1–3 of §1 we have "fundamental groups" in the usual sense. As abelian analogues, we have Grothendieck (or K-) groups. For $P=P(C)$ as in §1, such a universal group $G=G(C)$ would be crucial when we study analyticity, zeros, poles, and special values of zeta (and L-) functions in connection with spectral analysis on C; symbolically $Zet=Det$.

§3. Artin-Hecke type L-functions. Let F be a finite extension field of the rational number field $\boldsymbol{Q}$. We denote by O_F the integer ring. Let $P=P(O_F)$ be as in Example 1 (or 3) which is identified with the set of all maximal ideals of O_F. (In this case "maximal" is equivalent to "non-zero prime".) Then we have $d(P)=1$ and $\mu(P)=0$. We denote by $G=W(\bar{F}/F)$ the absolute Weil group of F. Let $\alpha: P \to \mathrm{Conj}\,(G)$ be any map such that $\alpha(p)$ contains a Frobenius element at (or, over) p (i.e., at $\boldsymbol{m}$ if $p=O_F/\boldsymbol{m}$) for each $p \in P$. (Our result is independent of the choice of α.) Let $E=(P, G, \alpha)$. Then we have

Theorem 1. *E is complete.*

This result was proved in a preprint "On the meromorphy of Euler products. Part II. Generalizations" cited in [1] and [2]. (In "Part III. Modifications", more general cases where representations were not necessarily unitary were treated; cf. [1, Remark 4].) That proof (which is rather long because of various complications) will be published in a series of papers, where we treat simultaneously other kinds of Euler products containing Euler products of Selberg type. As a corollary of Theorem 1, we have a solution of Linnik's problem: Theorem 1 of [2] holds if χ_i are unitary Grössencharacters without the assumption of finiteness of orders. (Cf. [2, Remark 3].)

Remark 2. An analogue of Linnik's problem is extended to a general fibre product $E_1 \underset{E_0}{\times \cdots \times} E_r$ of Euler data in a suitable sense.

References

[1] N. Kurokawa: On the meromorphy of Euler products. Proc. Japan Acad., 54A, 163–166 (1978).
[2] ——: On Linnik's problem. ibid., 54A, 167–169 (1978).