90. A Note on Recurrent Functions

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Abstract. We denote the set of recurrent functions and the set of distal functions by $RE(T, R^n)$ and $D(T, R^n)$, respectively. Then it is known ([1]) that $D(T, R^n)$ is a linear space, but that $RE(T, R^n)$ is not a linear space. The purpose of this paper is to strengthen the above results. We show that, if $f \in RE(T, R^n)-D(T, R^n)$, then there exist $g^1, g^2 \in RE(T, R^n)$ in its hull such that $g^1-g^2 \notin RE(T, R^n)$.

Let T denote real numbers R or integers Z. Let X be a metric space with the metric d_X. A continuous mapping $\pi: X\times T\to X$ is called a *flow on* (*a phase space*) X if π satisfies the following two conditions:

(1) $\pi(x, 0)=x$ for $x \in X$.

(2) $\pi(\pi(x, t), s)=\pi(x, t+s)$ for $x \in X$ and $t, s \in T$.

The orbit through $x \in X$ is denoted by $C_\pi(x)$. $M\subset X$ is called an *invariant set of* π if $C_\pi(x)\subset M$ for $x \in M$. The restriction of π to an invariant set $M\subset X$ is denoted by $\pi|M$. A non-empty compact invariant set M of π is called a *minimal set* if we have $\overline{C_\pi(x)}=M$ for every $x \in M$, where $\overline{C_\pi(x)}$ is closure of $C_\pi(x)$. If X is itself a minimal set of π, we say that π is a *minimal flow on* X. A flow π on X is said to be *equicontinuous* if for each $\varepsilon>0$ there exists a $\delta>0$ such that $d_X(\pi(x, t), \pi(y, t))<\varepsilon$ holds for $x, y \in X$ with $d_X(x, y)<\delta$ and for $t \in T$. A flow π on X said to be distal if $\inf_{t\in T}\{d_X(\pi(x, t), \pi(y, t))\}>0$ for each pair of distinct points $x, y \in X$. A point $x \in X$ is called an *almost automorphic point of* π if for each sequence $\{t_n\}\subset T$ there exists a subsequence $\{t_{n_k}\}\subset\{t_n\}$ such that $\pi(x, t_{n_k})\to y \in X$ and $\pi(y, -t_{n_k})\to x$ as $k\to\infty$ hold. A minimal flow is said to be *almost automorphic* if it contains an almost automorphic point. It is well known that every equicontinuous minimal flow on a compact metric space is distal and almost automorphic. Let π and ρ be flows on X and Y, respectively. A continuous mapping h of X into Y is called a homomorphism from π to ρ if we have $h(\pi(x, t))=\rho(h(x), t)$ for $(x, t) \in X\times T$.

Proposition 1. *Let π be a flow on a compact metric space X. If $x \in X$ is an almost automorphic point, then $\overline{C_\pi(x)}$ is a minimal set of π.*

Proof. If $\overline{C_\pi(x)}$ is not minimal, then there exists a minimal set $M\subset\overline{C_\pi(x)}$ such that $x \notin M$. Let $y \in M$. Then there exists a sequence $\{t_n\}\subset T$ such that $\pi(x, t_n)\to y$ as $n\to\infty$. Since x is an almost automorphic point of π, $\pi(y, -t_{n_k})\to x$ as $k\to\infty$ holds for some subsequence of $\{t_n\}$. But $\pi(y, -t_{n_k})\in M$ for all k, and hence we have $x\in M$. This is a contradiction. Hence $\overline{C_\pi(x)}$ is a minimal set of π.

Proposition 2. *Let π be a minimal flow on a compact metric space X. If π is almost automorphic but not equicontinuous, then π is not distal.*

Proof. If π is an almost automorphic, then there exists an equicontinuous minimal flow ρ on Y and a homomorphism h from π to ρ such that $h^{-1}(h(x_0))=\{x_0\}$ for some $x_0\in X$ (see [3] or [2]). Since π is not equicontinuous, there exists $x\in X$ such that $h^{-1}(h(x))\neq\{x\}$. If $x'\in h^{-1}(h(x))$ $(x'\neq x)$, then we have $\inf_{t\in T}\{d_X(\pi(x,t),\pi(x',t))\}=0$. This implies that π is not distal.

Let

$$C(T,R^n)=\{f: T\to R^n\ ;\ f \text{ is continuous}\}$$

with compact-open topology. Then $C(T,R^n)$ is a metric space. We denote a metric of it by d. Define a flow η on $C(T,R^n)$ by $\eta(f,t)=f_t$ for $(f,t)\in C(T,R^n)\times T$, where $f_t(s)=f(t+s)$ for $s\in T$. It is well known that it is well defined. The restriction of η to the hull $H(f)=\overline{\{f_t\}_{t\in T}}$ of $f\in C(T,R^n)$ by η_f. $f\in C(T,R^n)$ is said to be

(1) *recurrent* if $H(f)$ is compact and η_f is minimal,
(2) *almost periodic* if $H(f)$ is compact and η_f is equicontinuous,
(3) *distal* if $H(f)$ is compact and η_f is distal, and
(4) *almost automorphic* if $H(f)$ is compact and f is almost automorphic point of η.

Proposition 3. *Let π be a flow on a compact metric space X, and $\Phi: X\to R^n$ a continuous function. Define a mapping h from X into $C(T,R^n)$ by $h(x)=\Phi(\pi(x,\cdot))$ for $x\in X$. Then h is a homomorphism from π to η.*

Proof. Easy.

We denote the sets of recurrent functions, almost periodic functions, distal functions and almost automorphic functions by $RE(T,R^n)$, $AP(T,R^n)$, $D(T,R^n)$ and $AA(T,R^n)$, respectively.

Theorem. *If $f\in RE(T,R^n)-D(T,R^n)$, then there exist $g^1, g^2\in H(f)$ such that $g^1-g^2\notin RE(T,R^n)$.*

Proof. Since f is not distal, there exist $g^1, g^2\in H(f)$ $(g^1\neq g^2)$ such that $\inf_{t\in T}\{d(g^1_t, g^2_t)\}=0$. We consider the product flow $\eta_f\times\eta_f$ on $H(f)\times H(f)$ by

$$\eta_f\times\eta_f((h^1,h^2),t)=(h^1_t,h^2_t)$$

for $h^1, h^2\in H(f)$ and $t\in T$. Define $\Phi': H(f)\to R^n$ by $\Phi'(g)=g(0)$ for $g\in H(f)$. Then Φ' is continuous on $H(f)$. Define a mapping $\Phi: H(f)\times H(f)\to R^n$ by $\Phi(h^1,h^2)=\Phi'(h^1)-\Phi'(h^2)$ for $(h^1,h^2)\in H(f)\times H(f)$. Then Φ is also continuous on $H(f)\times H(f)$. By Proposition 3, Φ induces a homomorphism h from $\eta_f \times \eta_f$ to η. By the definition

$$\begin{aligned} h(h^1, h^2)(t) &= \Phi(\eta_f \times \eta_f((h^1, h^2), t))) \\ &= \Phi(h_t^1, h_t^2) = \Phi'(h_t^1) - \Phi'(h_t^2) \\ &= h^1(t) - h^2(t) \end{aligned}$$

for $(h^1, h^2) \in H(f) \times H(f)$ and $t \in T$. Since $\inf_{t \in T} \{(g_t^1, g_t^2)\} = 0$, there exist a sequence $\{t_n\} \subset T$ and $g \in H(f)$ such that $g_{t_n}^1 \to g$ and $g_{t_n} \to g$ as $n \to \infty$. Since every orbit closure is invariant and $\overline{C_{\eta_f \times \eta_f}((g^1, g^2))} \ni (g, g)$, we have $\overline{C_{\eta_f \times \eta_f}((g^1, g^2))} \supset \varDelta$, where $\varDelta$ is the diagonal set of $H(f) \times H(f)$. By continuity of h, we have

$$h(\overline{C_{\eta_f \times \eta_f}((g^1, g^2))}) = \overline{C_\eta(h(g^1, g^2))} = \overline{C_\eta(g^1 - g^2)}.$$

Hence $\overline{C_\eta(g^1 - g^2)}$ contains the 0-function k (i.e. $k(t) \equiv 0$), because the image of every element of $\varDelta$ by h is k. Hence $\eta_{g^1 - g^2}$ is not minimal. This implies that $g^1 - g^2 \notin RE(T, R^n)$.

Example. We consider the function f on Z defined by

$$f(n) = \operatorname{sgn}(\cos(2\pi\alpha n)) = \begin{cases} 1 & \cos(2\pi\alpha n) > 0 \\ -1 & \cos(2\pi\alpha n) < 0 \end{cases}$$

for $n \in Z$, where α is an irrational number. Then f is almost automorphic but not equicontinuous (see [3] p. 720). Hence f is not distal by Proposition 2. $H(f)$ contains following functions: Let $\cos(2\pi(m\alpha + x)) = 0$ for some $m \in Z$ and some $x \in [0, 1)$. Put

$$f^{x+m}(n) = \begin{cases} 1 & n = m \\ \operatorname{sgn}(\cos(2\pi(n\alpha + x))) & n \neq m \end{cases}$$

and

$$f^{x-m}(n) = \begin{cases} -1 & n = m \\ \operatorname{sgn}(\cos(2\pi(n\alpha + x))) & n \neq m \end{cases}$$

Then $f^{x+m}, f^{x-m} \in H(f)$, and hence $f^{x+m}, f^{x-m} \in RE(Z, R)$. By the definition of f^{x+m} and f^{x-m} we have

$$(f^{x+m} - f^{x-m})(n) = \begin{cases} 2 & n = m \\ 0 & n \neq m. \end{cases}$$

Hence $f^{x+m} - f^{x-m} \notin RE(Z, R)$.

References

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