89. *An Example of Coherent Singular Homology Groups*

By Akira KOYAMA
Department of Mathematics, Osaka Kyoiku University

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1. Lisica and Mardešić [4] developed the coherent prohomotopy category $\mathcal{CPHT}op$ and ANR-resolutions, and defined the strong shape theory $\mathcal{SSh}$ for arbitrary spaces. In [2], the author defined coherent singular homology groups of inverse systems and showed them invariants in $\mathcal{CPHT}op$. Then coherent singular homology groups of a space are defined as coherent singular homology groups of any one of its ANR-resolutions [6]. Hence coherent singular homology is actually a functor on $\mathcal{SSh}$. The purpose of this note is to show an existence of a 2-dimensional pointed movable continuum X in R^3 with $H_3^c(X:Q)\neq 0$. By the example, we see that coherent singular homology is different from Steenrod-Sitnikov and Čech ones.

2. In this note we consider only inverse systems of topological spaces and maps $\boldsymbol{X}=(X_a, p_{aa'}, A)$ over directed cofinite sets. By a *coherent map* $f:\boldsymbol{X}\to\boldsymbol{Y}=(Y_b, q_{bb'}, B)$, we mean an increasing function $\varphi: B\to A$ and a collection of maps $f_{\boldsymbol{b}}: \Delta^n\times X_{\varphi(b_n)}\to Y_{b_0}$, $\boldsymbol{b}=(b_0,\cdots,b_n)\in B^n$, $n\geq 0$, satisfying

$$(1)\qquad f_{\boldsymbol{b}}(\partial_j^n(t), x)=\begin{cases} q_{b_0b_1}f_{\boldsymbol{b}_0}(t,x) & \text{if } j=0,\\ f_{\boldsymbol{b}_j}(t,x) & \text{if } 0<j<n,\\ f_{\boldsymbol{b}_n}(t, p_{\phi(b_{n-1})\phi(b_n)}(x)) & \text{if } j=n,\end{cases}$$

where $x\in X_{\phi(b_n)}$, $t\in\Delta^{n-1}$, $n>0$,

$$(2)\qquad f_{\boldsymbol{b}}(\sigma_j^n(t), x)=f_{\boldsymbol{b}^j}(t,x),\qquad 0\leq j\leq n,$$

where $x\in X_{\phi(b_n)}$, $t\in\Delta^{n+1}$, $n\geq 0$.

here B^n, $n\geq 0$, denotes the set of all increasing sequences $\boldsymbol{b}=(b_0,\cdots,b_n)$ from B, and $\boldsymbol{b}_j=(b_0,\cdots,b_{j-1},b_{j+1},\cdots,b_n)$ and $\boldsymbol{b}^j=(b_0,\cdots,b_j,b_j,\cdots,b_n)$ for $0\leq j\leq n$. Each Δ^n, $n\geq 0$, is the standard n-simplex and $\partial_j^n:\Delta^{n-1}\to\Delta^n$, $\sigma_j^n:\Delta^{n+1}\to\Delta^n$ are the usual face and degeneracy operators, respectively. A *coherent homotopy* from f to f' is a coherent map $F:\boldsymbol{X}\times I=(X_a\times I, p_{aa'}\times 1, A)\to\boldsymbol{Y}$, given by $\Phi\geq\phi,\phi'$, and $F_{\boldsymbol{b}}$ such that

$$(3)\qquad F_{\boldsymbol{b}}(t,x,0)=f_{\boldsymbol{b}}(t, p_{\phi(b_n)\Phi(b_n)}(x)),$$
$$F_{\boldsymbol{b}}(t,x,1)=f'_{\boldsymbol{b}}(t, p_{\phi'(b_n)\Phi(b_n)}(x)),\qquad \text{where } x\in X_{\Phi(b_n)},\ t\in\Delta^n,\ n\geq 0.$$

Next, we define the *composition* gf of f and $g:\boldsymbol{Y}\to\boldsymbol{Z}=(Z_c, r_{cc'}, C)$. In the case that both $\boldsymbol{X}$ and $\boldsymbol{Y}$ are rudimentary systems (X) and (Y), and f is a map from X to Y, we define

$$(4)\qquad (gf)_{\boldsymbol{c}}(t,x)=g_{\boldsymbol{c}}(t, f(x)),\qquad \text{where } x\in X,\ t\in\Delta^n,\ \boldsymbol{c}\in C^n.$$

To define composition in the other case, one decomposes Δ^n into subpolyhedra $P_i^n=\{(t_0, \cdots, t_n)\in \Delta^n \mid t_0+\cdots+t_{i-1}\leq 1/2\leq t_0+\cdots+t_i\}$, $0\leq i \leq n$, and considers maps $\alpha_i^n: P_i^n\to\Delta^{n-i}$, $\beta_i^n: P_i^n\to\Delta^i$, where $\alpha_i^n(t)=(\sharp, 2t_{i+1}, \cdots, 2t_n)$, $\beta_i^n(t)=(2t_0, \cdots, 2t_{i-1}, \sharp)$, $\sharp=1-$sum of remaining terms. Then

(5) $(gf)_{\boldsymbol{c}}(t, x)=g_{c_0,\dots,c_i}(\beta_i^n(t), f_{\psi(c_i),\dots,\psi(c_n)}(\alpha_i^n(t), x))$,
where $\boldsymbol{c}=(c_0, \cdots, c_n)\in C^n$, $n\geq 0$, $x\in X_{\phi\psi(c_n)}$, $t\in P_i^n$, $0\leq i\leq n$
(see [4], § I.2).

The *identity coherent map* $1_X: X\to X$ is given by 1_A and $1_{\boldsymbol{a}}(t, x)=p_{a_0a_n}(x)$ for $\boldsymbol{a}=(a_0, \cdots, a_n)\in A^n$, $n\geq 0$, $x\in X_{a_n}$, $t\in\Delta^n$.

Lisica and Mardešić [4] showed that inverse systems of spaces and maps over directed cofinite sets and coherent homotopy classes of coherent maps construct a category. They call the category the *coherent prohomotopy category* and denote it $\mathcal{CPHT}op$. Our definition of composition is slightly different from the original one in [4], but by [4], Lemma I.9.7, we have the same category (cf. [2]).

Similarly, considering inverse systems of pointed spaces and base point preserving maps, we have the *pointed coherent prohomotopy category*, which is denoted by $\mathcal{CPHT}op_0$.

3. For an object X of $\mathcal{CPHT}op$, $S_i(X)$, $i\geq 0$, is the set of all coherent maps from Δ^i to X. Functions $d_k: S_i(X)\to S_{i-1}(X)$ and $s_k: S_{i+1}(X)\to S_i(X)$ are induced by ∂_k^i and σ_k^i, respectively. Then the triple $(S_i(X), d_k, s_k)$ induces a Kan complex $S_c(X)$, which is called the *coherent singular complex* of X. For a coherent map $f: X\to Y$, a semi-simplicial map $S_c(f): S_c(X)\to S_c(Y)$ is given by $S_i(f)(h)=fh$, $h\in S_i(X)$, $i\geq 0$. Now for an abelian group G, we define the i-th *coherent singular homology group* of X with the coefficient group G by

(6) $$H_i^c(X: G)=H_i(S_c(X): G).$$

A coherent map $f: X\to Y$ induces a homomorphism $f_*: H_i^c(X: G)\to H_i^c(Y: G)$ defined by

(7) $$f_*=S_c(f)_*.$$

Then by [2], coherent singular homology is a functor on $\mathcal{CPHT}op$. We note that if X is a rudimentary system (X), $S_c(X)$ and $H_i^c(X: G)$ are naturally isomorphic to the usual singular complex $S(X)$ and the usual singular homology groups $H_i(X: G)$, respectively ([2], Proposition 3.1 (2)).

Similarly, we can define *coherent singular cohomology groups* $H_c^i(X: G)$ of X by $H^i(S_c(X): G)$, and have the analogous properties.

4. Let $(X, x)=((X_a, x_a), p_{aa'}, A)$ be an object of $\mathcal{CPHT}op_0$. By $\pi_i^c(X, x)$, $i\geq 0$, we denote the set of all coherent homotopy classes of coherent maps $(S^i, s_0)\to(X, x)$, which is called the i-th *coherent prohomotopy group* of (X, x) (see [2] for the basic properties of coherent prohomotopy groups). Then we have the *Hurewicz homomorphism* $\Phi: \pi_i^c(X, x)\to H_i^c(X: Z)$ by

(8) $\Phi(\langle f\rangle)=f_*(1_{H_i(S^i:Z)})$, where $X=(X_a, p_{aa'}|X_{a'}, A)$ and $\langle f\rangle$ is the coherent homotopy class of a coherent map f.

The homomorphism $\Phi_Q:\pi_i^c(X, x)\to H_i^c(X:Q)$ is defined by the composition

$$\pi_i^c(X, x)\xrightarrow{\Phi} H_i^c(X:Z)\longrightarrow H_i^c(X:Q),$$

where the latter coefficient homomorphism is induced by the inclusion $Z\to Q$.

By [2], Theorem 5.2 and [1], Lemma 1, we have the key lemma of this note as follows;

Lemma. *Suppose that* $\pi_0^c(X, x)=\pi_1^c(X, x)=0$. *Then for* $\alpha\in\pi_q^c(X, x)$, *where* $q>1$, $\Phi_Q(\alpha)=0$ *if and only if there exist a pointed finite polyhedron* (K, k) *of* $\dim K<q$ *and a coherent map* $f:(K, k)\to(X, x)$ *such that* $\alpha\in f_\#(\pi_q(K, k))$.

5. Let $(A_n, *)$ and $(B_n, *)$, $n=1, 2, 3, \cdots$, be simply connected compact ANRs with base points satisfying the followings;

(i) if $n\neq m$, then $A_n\cap A_m=\{*\}=B_n\cap B_m$, and

(ii) $(\bigcup_{n\geq 1}A_n)\cap(\bigcup_{n\geq 1}B_n)=\{*\}$.

For each $n\geq 1$, define the simply connected compact ANR

$$(X_n, *)=((A_1, *)\vee(B_1, *))\vee\cdots\vee((A_n, *)\vee(B_n, *)),$$

and the map $p_{n,n+1}:(X_{n+1}, *)\to(X_n, *)$ by

$$p_{n,n+1}(x)=x \text{ for } x\in X_n, \text{ and } p_{n,n+1}(x)=* \text{ for } x\in A_{n+1}\vee B_{n+1}.$$

Thus, we have an inverse sequence $(X, *)=(X_n, p_{n,n+1})$ of simply connected compact ANRs. Since $(X, *)$ is movable, by [5] and [3], [7], the homomorphism $\eta:\pi_q^c(X, *)\to\check{\pi}_q(X, *)=\varprojlim \text{pro-}\pi_q(X, *)$ defined by

$$\eta(\langle f\rangle)=([f_n]) \quad \text{for } \langle f\rangle\in\pi_q^c(X, *),\ q\geq 0,$$

is an isomorphism, here $[f_n]$ is the homotopy class of the map f_n.

Let $i, j>1$ be fixed integers and let $q=i+j-1$. We will use the following notation: for $\alpha\in\pi_i(Y, y)$ and $\beta\in\pi_j(Y, y)$, $[\alpha, \beta]\in\pi_q(Y, y)$ is the *Whitehead product* of α and β. For every $n\geq 1$, let $\alpha_n\in\pi_i(A_n, *)$, $\beta_n\in\pi_j(B_n, *)$ and $\gamma_n=[\alpha_1, \beta_1]+\cdots+[\alpha_n, \beta_n]\in\pi_q(X_n, *)$. Then $(\gamma_n)\in\check{\pi}_q(X, *)$, and there is the unique coherent homotopy class $\gamma\in\pi_q^c(X, *)$ such that $\eta(\gamma)=(\gamma_n)$. By the analogous way of [1], Theorem 2, we have the following;

Theorem. $\Phi_Q(\gamma)\neq 0$ *if* $\phi_Q(\alpha_n)\neq 0$ *and* $\phi_Q(\beta_n)\neq 0$ *for infinitely many* $n\geq 1$, *where* ϕ_Q *is the composition* $\pi_k(Y, y)\to H_k(Y:Z)\to H_k(Y:Q)$ *of the Hurewicz homomorphism and the coefficient homomorphism induced by the inclusion* $Z\to Q$.

6. Coherent singular homology group $H_i^c(X:G)$ of a space X is defined by $H_i^c(X:G)$, where $p:X\to X$ is an ANR-resolution of X (see [6]). Let $F:X\to Y$ be a strong shape morphism given by a triple $(p, q, \langle f\rangle)$, where p, q are ANR-resolutions of X, Y and $f:X\to Y$ is a coherent map (see [4]). Then induced homomorphisms $F_*:H_i^c(X:G)$

$\to H_i^c(Y:G)$ are defined by f_*.

Similarly, for a pointed space (X, x) and a pointed strong shape morphism $F:(X, x)\to(Y, y)$, we can define coherent prohomotopy groups $\pi_i^c(X, x)$ and induced homomorphisms $F_\#:\pi_i^c(X, x)\to\pi_i^c(Y, y)$.

Example. Let $k>1$ be a fixed integer. For each $n\geq 1$, let

$$S(k, n)=\{(x_1, \cdots, x_{k+1})\in R^{k+1} \mid (x_1-(1/n))^2+x_2^2+\cdots+x_{k+1}^2=(1/n^2)\}.$$

Put

$$S(k)=\bigcup_{n\geq 1} S(k, n) \quad \text{and} \quad s_k=(0, \cdots, 0)\in S(k).$$

Then $(S(k), s_k)$ is a k-dimensional pointed movable continuum in R^{k+1}. By Theorem, we have that

$$H_{2k-1}^c(S(k):Q)\neq 0.$$

Particularly, the continuum $S(2)$ satisfies the condition required in 1.

References

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