# 86. A Stability Theorem on the Boundary Identification for Coefficients of Hyperbolic Equations

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In this note, we consider the hyperbolic equation

(1) $\partial_t^2 u+(-\partial_x^2+p(x))u=0 \quad (0<x<1,\ -\infty<t<\infty)$

with the boundary condition

(2) $(-\partial_x+h)u|_{x=0}=(\partial_x+H)u|_{x=1}=0 \quad (-\infty<t<\infty)$

and with the initial condition

(3) $u|_{t=0}=a_0(x),\quad \partial_t u|_{t=0}=a_1(x) \quad (0<x<1).$

We suppose that the coefficients $P\equiv(p, h, H)\in C_x^0[0,1]\times \boldsymbol{R}\times\boldsymbol{R}$ and the initial values $a\equiv(a_0, a_1)\in H_x^1(0,1)\times L_x^2(0,1)$ are unknown, while the boundary values of the solution $u=u(x,t)\in C_t^0((-\infty,\infty)\to H_x^1(0,1))\cap C_t^1((-\infty,\infty)\to L_x^2(0,1))\subset C_{x,t}^0((-\infty,\infty)\times[0,1])$

(4) $u|_{x=0}=f_0(t) \quad u|_{x=1}=f_1(t) \quad (-T\leqq t\leqq T)$

are observed and known for some $T>0$. In order to study the identifiability (see [2], e.g.), let us consider the model equation

(5) $\partial_t^2 v+(-\partial_x^2+q(x))v=0 \quad (0<x<1,\ -\infty<t<\infty)$

with

(6) $(-\partial_x+j)v|_{x=0}=(\partial_x+J)v|_{x=1}=0 \quad (-\infty<t<\infty)$

and

(7) $v|_{t=0}=b_0(x),\quad \partial_t v|_{t=0}=b_1(x) \quad (0<x<1)$

for $Q\equiv(q, j, J)\in C_x^0[0,1]\times\boldsymbol{R}\times\boldsymbol{R}$ and $b\equiv(b_0, b_1)\in H_x^1(0,1)\times L_x^2(0,1)$. Then, the functions

(8) $\varepsilon_0(t)=g_0(t)-f_0(t),\quad \varepsilon_1(t)=g_1(t)-f_1(t) \quad (-T\leqq t\leqq T)$

stand for the errors of identification, where

(9) $g_0(t)=v|_{x=0},\quad g_1(t)=v|_{x=1} \quad (-T\leqq t\leqq T).$

To state our results, we introduce the following:

**Notation 1.** $A_{p,h,H}$ denotes the Sturm-Liouville operator $-\partial_x^2+p(x)$ in $L_x^2(0,1)$ with the boundary condition

$$(-\partial_x+h)\cdot|_{x=0}=(\partial_x+H)\cdot|_{x=1}=0.$$

**Notation 2.** $\sigma(A_{p,h,H})=\{\lambda_n\}_{n=0}^{\infty}\ (-\infty<\lambda_0<\lambda_1<\cdots\to\infty)$ and $\{\phi\}_{n=0}^{\infty}$ denote the eigenvalues and the eigenfunctions of $A_{p,h,H}$, respectively, the latter being normalized by $\|\phi_n\|_{L_x^2(0,1)}=1$.

**Notation 3.** The equation (1) with (2)–(3) is denoted by $E(P,a)$.

**Definition 1.** We say $E(P,a)\in G$ if

(10) $(a_n^0)^2+(a_n^1)^2\neq 0$

for any $n\in\boldsymbol{N}\equiv\{0,1,2,\cdots\}$, where

(11) $$a_n^0=(a_0, \phi_n)_{L_x^2(0,1)}, \qquad a_n^1=(a_1, \phi_n)_{L_x^2(0,1)}.$$

We have the following theorem on the uniqueness of the problem:

**Theorem 1.** *If $E(P, a) \in G$, $T \geqq 2$ and*

(12) $$\varepsilon_0(t)=\varepsilon_1(t)=0 \qquad (-T \leqq t \leqq T),$$

*then*

(13) $$(Q, b)=(P, a)$$

*follows. If $E(P, a) \notin G$, conversely, there exists $Q=(q, j, J) \in C_x^0[0, 1] \times \boldsymbol{R} \times \boldsymbol{R}$ and $b \equiv (b_0, b_1) \in H_x^1(0, 1) \times L_x^2(0, 1)$ such that*

(12′) $$\varepsilon_0(t)=\varepsilon_1(t)=0 \qquad (-\infty < t < \infty),$$

*in spite of*

(13′) $$Q \neq P.$$

For the proof, see [7].

We now want to show the following theorem on the stability of the problem:

**Definition 2.** For $\alpha>1/2$, we say $E(P, a) \in G_\alpha$ if $p \in C_x^\alpha[0, 1]$ and

(14) $$M_\alpha^{-1}(n^2+1)^{-\alpha} \leqq (n^2+1)(a_n^0)^2+(a_n^1)^2 \leqq M_\alpha(n^2+1)^{-\alpha} \qquad (n \in \boldsymbol{N})$$

for some constant $M_\alpha>0$.

Here, $\alpha>1/2$ is a compatibility condition for (14) to $a=(a_0, a_1) \in H_x^1(0, 1) \times L_x^2(0, 1)$. Note the relation

$$D((A_{p,h,H}+\lambda)^{1/2})=H_x^1(0, 1) \qquad \text{for } \lambda>-\lambda_0$$

and the asymtotic formula

(15) $$\lambda_n^{1/2}=n\pi+O(1/n) \qquad (n \to \infty).$$

**Theorem 2.** *If $E(P, a) \in G_\alpha$ and $T \geqq 2$, then we have*

(16) $$\|q-p\|_{L_x^2(0,1)}+|j-h|+|J-H| \leqq c(\kappa)\{\|\varepsilon_0\|_{H_t^{\alpha+2}(-T,T)}+\|\varepsilon_1\|_{H_t^{\alpha+2}(-T \leqq t \leqq T)}\}$$

*for each $\kappa>0$, provided that*

(17) $$\|q\|_{L_x^2(0,1)}+|j|+|J| \leqq \kappa.$$

*The constant $c(\kappa)$ in (16) depends only on $\|p\|_{C_x^\alpha[0,1]}$, $|h|$, $|H|$, $T \geqq 2$, $\alpha>1/2$, $M_\alpha$ in (14), and $\kappa>0$.*

By a theorem on the non-harmonic Fourier series [3], we see that

(18) $$f_0, f_1 \in H_t^\beta(-T, T) \qquad \text{for } 0 \leqq \beta < \alpha+1/2$$

and

(19) $$f_0, f_1 \notin H_t^\beta(-T, T) \qquad \text{for } \alpha+1/2<\beta,$$

follow from (14), if $p$ is sufficiently smooth. On the other hand, the norms $\|\varepsilon_0\|_{H_t^{\alpha+2}(-T,T)}$ and $\|\varepsilon_1\|_{H_t^{\alpha+2}(-T,T)}$ in the right side of (16) are best possible. The gap $(\alpha+2)-(\alpha+1/2-\varepsilon)=3/2+\varepsilon$ ($\varepsilon>0$) of these Sobolev exponents shows the ill-posedness of the present problem. Also it suggests an important role of the irregularity of data in such an identification. Actually, the inequality (16) owes much to the property of the finite propagation of the hyperbolic equation (1). Therefore, we cannot expect even such stability theorems as Theorem 2, for the parabolic inverse problems studied in [9] and [4]–[6].

Our problem is related to the work [1]. For more details, see [8].

## References


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