83. *α-additive Functions and Uniform Distribution modulo One*

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1. Throughout this note, we write $e(x)=e^{2\pi i x}$ for real x and denote by N_0 the set of all nonnegative integers. Let α be an irrational number and $[a_0; a_1, \cdots, a_k, \cdots]$ be the continued fraction expansion of α. The sequence $\{q_k\}$ of denominators of convergents for α satisfies

$$q_0=1,\ q_1=a_1 \quad \text{and} \quad q_{k+2}=a_{k+2}q_{k+1}+q_k \qquad \text{for all } k\in N_0.$$

Every nonnegative integer can be written in the form

$$n=\sum_{k=0}^{\infty}\varepsilon_k(n)q_k,$$

where

$$\varepsilon_0(n)\in\{0,1,\cdots,a_1-1\},$$
$$\varepsilon_k(n)\in\{0,1,\cdots,a_{k+1}\},$$

and for $k\geqq 1$ $\varepsilon_{k-1}(n)=0$ whenever $\varepsilon_k(n)=a_{k+1}$. This representation is unique.

Definition. A function (or a sequence) $f: N_0\to \boldsymbol{R}$ is said to be α-additive if $f(0)=0$ and

$$f(n)=\sum_{k=0}^{\infty}f(\varepsilon_k(n)q_k).$$

J. Coquet [1] showed that the α-additive sequence

$$\{\sigma_\alpha(n)\}=\{x\sum_{k=0}^{\infty}\varepsilon_k(n)\}$$

is uniformly distributed modulo one (abbreviated: u.d. mod 1) if and only if x is irrational. In this note, we prove the following theorem which gives a generalization of this result of J. Coquet's.

Theorem. *Let $\phi: N_0\to \boldsymbol{R}$ be a function with $\phi(0)=0$. We set*

$$f(n)=\sum_{k=0}^{\infty}\phi(\varepsilon_k(n)).$$

If $\phi(1)$ is irrational and the sequence $\{\phi(n)\}_{n\in N_0}$ is u.d. mod 1, *then the sequence $\{f(n)\}_{n\in N_0}$ is u.d.* mod 1.

Immediate consequences of this theorem will be the following:

Corollary 1. *Let $\{a_k\}$ be an unbounded sequence, and $\phi(n)$ and $f(n)$ be the functions given in the theorem. If $\{\phi(n)\}$ is u.d.* mod 1, *then $\{f(n)\}$ is u.d.* mod 1.

Corollary 2. *Let $\{a_k\}$ be a bounded sequence, and $\phi(n)$ and $f(n)$ be as in the theorem. If $\phi(1)$ is irrational, then $\{f(n)\}$ is u.d.* mod 1.

Corollary 3. *Let $\{a_k\}$ be bounded and assume that $a_k\geqq 3$ for infinitely many k. Let $\phi(n)$ and $f(n)$ be as in the theorem. If $\phi(1)$ is rational and $\phi(2)$ is irrational, then $\{f(n)+x\sigma_\alpha(n)\}$ is u.d.* mod 1 *for any real x.*

2. We set

$$\mu_k=\frac{1}{q_k}\sum_{n<q_k} e(f(n)).$$

To prove our theorem, we need the following lemma due to J. Coquet [1].

Lemma. *Let $f(n)$ be an α-additive function. Then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n<N} e(f(n))=0$$

if and only if $\lim_{k\to\infty}|\mu_k|=0$.

Proof of Theorem. Now, $f(n)$ being assumed to be α-additive, we have

$$\begin{aligned}\mu_{k+1}q_{k+1}&=\sum_{b<q_k}\sum_{a<a_{k+1}} e(f(aq_k+b))+\sum_{b<q_k} e(f(a_{k+1}q_k+b))\\&=\mu_k q_k\sum_{a<a_{k+1}} e(\phi(a))+\mu_{k-1}q_{k-1}e(\phi(a_{k+1}))\end{aligned}$$

for every integer $k\geqq 1$, and

$$\begin{aligned}\mu_{k+2}q_{k+2}&=\mu_{k+1}q_{k+1}\sum_{b<a_{k+2}} e(\phi(b))+\mu_k q_k e(\phi(a_{k+2}))\\&=\mu_k q_k((\sum_{b<a_{k+2}} e(\phi(b)))(\sum_{a<a_{k+1}} e(\phi(a)))+e(\phi(a_{k+2})))\\&\quad+\mu_{k-1}q_{k-1}e(\phi(a_{k+1}))\sum_{b<a_{k+2}} e(\phi(b))\end{aligned}$$

for every integer $k\geqq 2$.

If we put

$$M_k=\max\{|\mu_k|, |\mu_{k-1}|\}\qquad \text{for } k=1, 2, \cdots,$$

then

$$(1)\qquad |\mu_{k+1}|<M_k\left(|\sum_{a<a_{k+1}} e(\phi(a))|\frac{q_k}{q_{k+1}}+\frac{q_{k-1}}{q_{k+1}}\right)=M_kA_k,\quad \text{say},$$

$$(2)\qquad |\mu_{k+2}|<M_k\left((|\sum_{b<a_{k+2}} e(\phi(b))||\sum_{a<a_{k+1}} e(\phi(a))|+1)\frac{q_k}{q_{k+2}}+|\sum_{b<a_{k+2}} e(\phi(b))|\frac{q_{k-1}}{q_{k+2}}\right)=M_kB_k,\quad \text{say}.$$

It follows from (1) and (2) that

$$(3)\qquad M_{k+2}\leqq M_k\max\{A_k, B_k\}.$$

First we assume that $\{a_k\}$ is unbounded. Then there is a strictly (and indefinitely) increasing sequence $\{a_{k_j}\}$ of $\{a_k\}$ and we have for this sequence $\{a_{k_j}\}$

$$A_{k_j-1}\leqq\frac{1}{a_{k_j}}|\sum_{a<a_{k_j}} e(\phi(n))|+\frac{1}{a_{k_j}},$$

and

$$B_{k_j-1}\leqq\frac{1}{a_{k_j}}|\sum_{a<a_{k_j}} e(\phi(n))|+\frac{2}{a_{k_j}}.$$

It is easy to see from (1) that the sequence $\{M_k\}$ is decreasing, and so the limit

$$c=\lim_{k\to\infty} M_k$$

exists. Since by assumption $\{\phi(n)\}$ is u.d. mod 1, we must have $c=0$, by (3).

Secondly, we assume that $\{a_k\}$ is bounded. Let L be an upper bound of $\{a_k\}$. The case that $a_k=1$ for all sufficiently large k has been treated by J. Coquet [1], and therefore we may assume that there are infinitely many k such that $a_k>1$. We take a subsequence $\{a_{k_j}\}$ of $\{a_k\}$ such that $a_{k_j}=d>1$ where d is a number independent of k_j. Since $\phi(1)$ is irrational, we have

$$\delta=d-\left|\sum_{n<d} e(\phi(n))\right|>0.$$

Then we find

$$A_{k_j-1} \leqq (a_{k_j}-\delta)\frac{q_{k_j-1}}{q_{k_j}}+\frac{q_{k_j-2}}{q_{k_j}}=1-\frac{q_{k_j-1}}{q_{k_j}}\delta<1-\frac{\delta}{L+1}$$

and

$$\begin{aligned} B_{k_j-1} &\leqq (1+a_{k_j+1}(a_{k_j}-\delta))\frac{q_{k_j-1}}{q_{k_j+1}}+a_{k_j}\frac{q_{k_j-2}}{q_{k_j+1}} \\ &=1-\frac{a_{k_j+1}q_{k_j-1}}{q_{k_j+1}}\delta<1-\frac{\delta}{(L+1)^2} \end{aligned}$$

which implies by (3) that the limit

$$c=\lim_{k\to\infty} M_k$$

satisfies

$$c\leqq c(1-\delta(L+1)^{-2}).$$

Thus we have $c=0$, and the proof is complete.

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Reference

[1] J. Coquet: Répartition de la somme des chiffres associée à une fraction continue. Bull. Soc. Roy. Sci. Liège, **51**, 161–165 (1982).