82. On Existence of the Solution of a Certain Integral Equation of the First Kind in the sense of a Distribution

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§1. Let L be a union of open arcs introduced in the previous paper [3], and consider an integral equation of Fredholm of the first kind defined on L;

$$(1)\qquad \Psi\tau \equiv \int_L \psi(x, y)\tau(y)ds_y = g(x), \qquad x \in L,$$

where the kernel $\psi(x, y)$ has been defined in [3] in terms of the zero-th order Hankel function of the second kind. (The reader is referred to [3] for the definitions of L and ψ as well as other notations employed in the present paper.)

The equation (1) has been tightly related to the "Dirichlet problem for the Helmholtz equation for an open boundary L", which has been defined as the problem of finding a function $u(x)$ which (i) satisfies the Helmholtz equation at points $x \in \boldsymbol{R}^2 - \bar{L}$, (ii) assumes given boundary values when the point x tends to a point on L from the positive as well as negative sides of L, (iii) satisfies the radiation condition at infinity, and (iv) satisfies the edge condition in the vicinity of end points x^* of L. The detailed definition of the problem is found in [1]. Also, in [1], the following theorems have been obtained.

Theorem 1. *A solution of the Dirichlet problem, if exists, is necessarily given by*

$$(2)\qquad u(x) = \int_L \psi(x, y)\tau(y)ds_y - u_0(x), \qquad x \in \boldsymbol{R}^2 - \bar{L},$$

where $u_0(x)$ is a known function given by the boundary data, and τ must satisfy the integral equation (1). *Conversely, if τ is a solution of* (1) *and if u is defined by* (2) *in terms of τ, then, u is the solution of the Dirichlet problem satisfying all requirements* (i)–(iv) *mentioned above. That is, the Dirichlet problem is equivalent to that of solving for the integral equation* (1).

Theorem 2. $\Psi\tau = 0 \rightleftarrows \tau = 0$.
That is, the solution of (1), *if exists, is unique.*

It is the purpose of this paper to prove the existence of the solution of the integral equation (1). In what follows, taking (1) in the sense of a distribution on $\mathcal{D} = C_0^\infty(\bar{L})$, we shall prove the existence of a solution τ in the sense of a functional on $\mathcal{E}=C^{\infty}(\bar{L})$.

§2. Theorem 3 shown below, which has been proved in [3], will play an important role in what follows.

Theorem 3. *For* $\forall\sigma\in\mathcal{D}$, *set* $\hat{\sigma}=\Psi\sigma$, *then*, $\hat{\sigma}\in\mathcal{E}$. *Furthermore, we have*

$$\hat{\sigma}\to 0 \text{ in } \mathcal{E} \quad \rightleftarrows \quad \sigma\to 0 \text{ in } \mathcal{D}.$$

With help of this theorem, we can prove

Lemma 1. *Set* $\hat{\Sigma}=\{\hat{\sigma}\,;\hat{\sigma}=\Psi\sigma,\ \sigma\in\mathcal{D}\}$, *then*, $\hat{\Sigma}$ *is a closed subset of* $\mathcal{E}$.

Proof. Assume that $\{\hat{\sigma}_m\}$ is a converging sequence in $\hat{\Sigma}$ such that $\|\hat{\sigma}_p-\hat{\sigma}_q\|\to 0$ in $\mathcal{E}$ holds when $p, q\to\infty$. Then, by virtue of Theorem 3, it follows that $\|\sigma_p-\sigma_q\|\to 0$ in $\mathcal{D}$, where $\hat{\sigma}_p=\Psi\sigma_p$ and $\hat{\sigma}_q=\Psi\sigma_q$. Hence, there exists an element $\sigma\in\mathcal{D}$ such that $\|\sigma_p-\sigma\|\to 0$ in $\mathcal{D}$ holds when $p\to\infty$. If we set $\hat{\sigma}=\Psi\sigma$, then, we can show that $\|\hat{\sigma}_p-\hat{\sigma}\|\leqq\|\Psi\|\,\|\sigma_p-\sigma\|$ and $\|\hat{\sigma}_p^{(m)}-\hat{\sigma}^{(m)}\|\leqq\|\Psi_m\|\,\|\sigma_p^{(m)}-\sigma^{(m)}\|$ hold, where we have set

$$\|\Psi\|=\sup\cdot\int_L|\psi(x, y)|\,ds_y<\infty$$

and

$$\|\Psi_m\|=\sup\cdot\int_L|\psi^{[m]}(x, y)|\,ds_y<\infty.$$

Consequently, we have $\hat{\sigma}_p\to\hat{\sigma}\in\hat{\Sigma}$ when $p\to\infty$. Thus, we have proved the Lemma.

Lemma 2. $\hat{\Sigma}$ *is dense in* $\mathcal{E}$.

Proof. Assume that the contrary is true. Then, by Hahn-Banach's extension theorem, there exists a functional $f\in\mathcal{E}'$, where $\mathcal{E}'$ is the space conjugate to $\mathcal{E}$, such that $f(\hat{\sigma})=0$ for $\forall\hat{\sigma}\in\hat{\Sigma}$ and that $f(\phi_1)=1$ for $\exists\phi_1\in\mathcal{E}-\hat{\Sigma}$. Again, by the Hahn-Banach theorem, f is extended to a functional on $L_2(\bar{L})$, which is, by F. Riesz's theorem, expressed as $f(\phi)=(\phi, \phi_2)$ with $\exists\phi_2\in L_2(\bar{L})$. As a consequence, it follows that

$$(\phi_2, \hat{\sigma})=\int_L \phi_2(x)\hat{\sigma}(x)ds_x=\int_L \sigma(x)ds_x\int_L \psi(x, y)\phi_2(y)ds_y=0$$

holds for $\forall\sigma\in\mathcal{D}$. Therefore, we have

$$\int_L \psi(x, y)\phi_2(y)ds_y=0,$$

which, by virtue of Theorem 2, implies $\phi_2(x)=0$. However, this contradicts to $f(\phi_1)=(\phi_2, \phi_1)=1$. (Q.E.D.)

Combining these two lemmas, we have

Theorem 4. $\hat{\Sigma}=\mathcal{E}$.

This theorem, together with Theorem 2, shows that the correspondence between $\sigma\in\mathcal{D}$ and $\hat{\sigma}=\Psi\sigma\in\mathcal{E}$ is one-to-one. As a consequence, we have

Theorem 5. *The integral equation* (1); $\Psi\tau=g$, *assumes always* *a unique solution* $\tau \in \mathcal{E}$ *corresponding to* $\forall g \in \mathcal{D}$.

Furthermore, we can prove

Theorem 6. (1); $\Psi\tau = g$ *assumes always a functional solution* $\tau \in \mathcal{E}'$ *for an arbitrarily given distribution* $g \in \mathcal{D}'$.

Proof. Note that a functional $\tau \in \mathcal{E}'$ can be defined by $g \in \mathcal{D}'$ in such a manner that $\tau(\hat{\sigma}) = g(\sigma)$. Rewrite it as $\tau(\hat{\sigma}) = \tau(\Psi\sigma) = \Psi^*\tau(\sigma)$, and we have $\Psi^*\tau = g$.

Finally, from Theorem 4, it is not difficult to prove the existence of the Green function $\gamma(x, x')$ for the open boundary L, and also the continuity dependence of the solution of the Dirichlet problem for L on the boundary data. Namely, we have the following theorems.

Theorem 7. *There exists a function* $\gamma(x, x')$, *where* $x \in L$ *and* $x' \in \boldsymbol{R}^2 - \bar{L}$, *such that* (i) *it belongs to* $\mathcal{D}(\bar{L})$ *with respect to* x, *and to* $C^\infty(\boldsymbol{R} - \bar{L})$ *with respect to* x', (ii) *it satisfies the Helmholtz equation with respect to* x', (iii) *it satisfies the radiation condition when* $|x'| \to \infty$, *and* (iv) $\gamma(x, x_0) = \delta(x, x_0)$ *when* $x' \to x_0 \in L$, *where* δ *is the Dirac's delta function.*

Proof. Obviously, $\psi(x, x')$ is infinitely many times differentiable with respect to x and x' as far as $x \neq x'$. That is, $\psi(x, x') \in \mathcal{E}_x$. Hence, by Theorem 4, $\psi(x, x') \in \hat{\Sigma}_x$, and there exists $\exists\gamma(x, x') \in \mathcal{D}_x$ such that

$$\psi(x, x') = \int_L \psi(x, y)\gamma(y, x')ds_y.$$

With help of this expression, and also by Theorem 2, it is not difficult to prove the theorem.

Corollary. *The solution* $u(x)$ *of the Dirichlet problem for the boundary L, accompanied by the boundary value* $g(x)$, *is expressed as*

$$u(x') = \int_L \gamma(y, x')g(y)ds_y \qquad \textit{for } x' \in \boldsymbol{R}^2 - \bar{L}.$$

Proof. This is proved by the property of $\gamma(x, x')$ mentioned above, and the uniqueness theorem 2.

Corollary. *The solution* $u(x')$ *depends on the boundary data continuously.*

Proof. From the expression for $u(x')$ mentioned above, we have $|u(x')| \leqq \|\gamma\| \|g\|$, where $\|g\| = \sup_{x \in L} |g(x)|$, while

$$\|\gamma\| = \sup_{x' \in D} \cdot \int_L |\gamma(y, x')| \, ds_y$$

where D is a compact subset of $\boldsymbol{R}^2 - \bar{L}$.

Full papers of this and the previous papers will appear in some journal.

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