80. *Microlocalization at Infinity of the Sheaf of Real Analytic Functions*

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The theory of the sheaf $\mathcal{C}^t$ described in [3] allows to precise the global behavior at infinity of functions which are real analytic over the whole space $\boldsymbol{R}^n$. Here, following a suggestion of A. Kaneko, we improve this theory by considering the direct image of $\mathcal{A}$ in the radial compactification of $\boldsymbol{R}^n$. This enables us to characterize the different kinds of decompositions in terms of holomorphic functions in conic tubes, which are admitted by a real analytic function near a single point at infinity in $\boldsymbol{D}^n$. The geometric methods of microlocalization described in [2] and [4] constitute again the basic tool to carry out the proofs.

As usual, we shall denote by $\boldsymbol{R}^n$ the n-dimensional euclidean space but to provide more simplicity in some formulas, we shall assume $n>1$, despite the fact that this theory may be carried out with slight modifications in the one-dimensional case. Let us then consider the following topological spaces: $\boldsymbol{D}^n$ the radial compactification of $\boldsymbol{R}^n$, $S\boldsymbol{R}^n$ (resp. $S\boldsymbol{D}^n$) the spherical normal bundle to $\boldsymbol{R}^n$ (resp. $\boldsymbol{D}^n$), $S^*\boldsymbol{R}^n$ (resp. $S^*\boldsymbol{D}^n$) the cospherical normal bundle to $\boldsymbol{R}^n$ (resp. $\boldsymbol{D}^n$), $\tilde{\boldsymbol{R}}^n$ (resp. $\tilde{\boldsymbol{D}}^n$), the real monoidal transform of $\boldsymbol{C}^n$ (resp. $\boldsymbol{D}^n+i\boldsymbol{R}^n$) with center $\boldsymbol{R}^n$ (resp. $\boldsymbol{D}^n$), $\tilde{\boldsymbol{R}}^{n*}$ (resp. $\tilde{\boldsymbol{D}}^{n*}$) the real comonoidal transform of $\boldsymbol{C}^n$ (resp. $\boldsymbol{D}^n+i\boldsymbol{R}^n$) with center $\boldsymbol{R}^n$ (resp. $\boldsymbol{D}^n$), $D\boldsymbol{R}^n$ (resp. $D\boldsymbol{D}^n$) the set $\{(x,\xi,\eta)\in S\boldsymbol{R}^n\times_{\boldsymbol{R}^n}S^*\boldsymbol{R}^n:\langle\xi,\eta\rangle\geq 0\}$ (resp. $\{(x,\xi,\eta)\in S\boldsymbol{D}^n\times_{\boldsymbol{D}^n}S^*\boldsymbol{D}^n:\langle\xi,\eta\rangle\geq 0\}$). We denote respectively by ι,α,β and γ the natural injections $\boldsymbol{C}^n\to\boldsymbol{D}^n+i\boldsymbol{R}^n$, $\tilde{\boldsymbol{R}}^n\to\tilde{\boldsymbol{D}}^n$ (or its restriction $S\boldsymbol{R}^n\to S\boldsymbol{D}^n$), $\tilde{\boldsymbol{R}}^{n*}\to\tilde{\boldsymbol{D}}^{n*}$ (or its restriction $S^*\boldsymbol{R}^n\to S^*\boldsymbol{D}^n$) and $D\boldsymbol{R}^n\to D\boldsymbol{D}^n$. Following [2], we denote also respectively by $\tau,\tau',\pi,\pi',\tau_.,\tau'_.,\pi_.,\pi'_.,\varepsilon'$ and $\hat{\varepsilon}'$ the natural projections $\tilde{\boldsymbol{R}}^n\to\boldsymbol{C}^n$ (or its restriction $S\boldsymbol{R}^n\to\boldsymbol{R}^n$), $\tilde{\boldsymbol{D}}^n\to\boldsymbol{D}^n+i\boldsymbol{R}^n$ (or its restriction $S\boldsymbol{D}^n\to\boldsymbol{D}^n$), $\tilde{\boldsymbol{R}}^{n*}\to\boldsymbol{C}^n$ (or its restriction $S^*\boldsymbol{R}^n\to\boldsymbol{R}^n$), $\tilde{\boldsymbol{D}}^{n*}\to\boldsymbol{D}^n+i\boldsymbol{R}^n$ (or its restriction $S^*\boldsymbol{D}^n\to\boldsymbol{D}^n$), $D\boldsymbol{R}^n\to S^*\boldsymbol{R}^n$, $D\boldsymbol{D}^n\to S^*\boldsymbol{D}^n$, $D\boldsymbol{R}^n\to S\boldsymbol{R}^n$, $D\boldsymbol{D}^n\to S\boldsymbol{D}^n$, $(\boldsymbol{D}^n+i\boldsymbol{R}^n)\backslash\boldsymbol{D}^n\to\boldsymbol{D}^n+i\boldsymbol{R}^n$ and $\tilde{\boldsymbol{D}}^n\backslash S\boldsymbol{D}^n\,[\simeq(\boldsymbol{D}^n+i\boldsymbol{R}^n)\backslash\boldsymbol{D}^n]\to\tilde{\boldsymbol{D}}^n$.

Let us denote respectively by $\mathcal{O},\mathcal{A},\mathcal{B},\mathcal{C}$ and $\mathcal{Q}$ the sheaves of germs of holomorphic functions over $\boldsymbol{C}^n$, of real analytic functions over $\boldsymbol{R}^n$, of hyperfunctions over $\boldsymbol{R}^n$, of microfunctions over $S^*\boldsymbol{R}^n$ and the cohomology sheaf $\mathcal{H}^1_{S\boldsymbol{R}^n}(\tau^{-1}\mathcal{O})$ introduced in [2] and [4]. The following two sheaves will also play a capital role in this theory:

$$\tilde{\mathcal{A}}' := (\tilde{\varepsilon}'_* \varepsilon'^{-1} \iota_* \mathcal{O})|_{SD^n},$$
$$\tilde{\mathcal{O}} := [(\alpha_* \tau^{-1} \mathcal{O})|_{SD^n}] / \tau'^{-1}[(\iota_* \mathcal{O})|_{D^n}].$$

On this point, it is interesting to notice that $\tilde{\mathcal{O}}$ "concentrates" all its informations over the "boundary" of D^n, i.e.: $\tilde{\mathcal{O}}|_{SD^n \setminus SR^n} = 0$.

Proposition 1. *With the above notations, we obtain for any k in Z:*

$$(R^k \iota_*)\mathcal{O} = \iota_* \mathcal{O}.\delta_{k,0}; \quad (R^k \tilde{\varepsilon}'_*)\varepsilon'^{-1}\iota_* \mathcal{O} = \tilde{\varepsilon}'_* \varepsilon'^{-1} \iota_* \mathcal{O}.\delta_{k,0};$$
$$(R^k \alpha_*)\tau^{-1}\mathcal{O} = \tau'^{-1}(R^k \iota_*)\mathcal{O} = \tau'^{-1}\iota_* \mathcal{O}.\delta_{k,0} = \alpha_* \tau^{-1} \mathcal{O}.\delta_{k,0};$$
$$\mathcal{H}^k_{SD^n}(\tau'^{-1}\iota_* \mathcal{O}) = \tilde{\mathcal{A}}' / \tau'^{-1}[(\iota_* \mathcal{O})|_{D^n}].\delta_{k,1};$$
$$(R^k \alpha_*)Q = \alpha_* Q.\delta_{k,0} = \tilde{\mathcal{A}}' / (\alpha_* \tau^{-1} \mathcal{O})|_{SD^n}.\delta_{k,0},$$

where $\delta_{k,j}$ ("multiplication" by Kronecker's symbol) means that the considered k-th derived functor vanishes for every $k \neq j$.

As there exists a commutative diagram of morphisms of sheaves

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \tau'^{-1}[(\iota_* \mathcal{O})|_{D^n}] & = & \tau'^{-1}[(\iota_* \mathcal{O})|_{D^n}] & & \\
 & & \downarrow & & \downarrow & & \\
0 \to & & (\alpha_* \tau^{-1} \mathcal{O})|_{SD^n} & \to & \tilde{\mathcal{A}}' & \to \alpha_* Q \to 0 & \\
 & & \downarrow & & \downarrow & \parallel & \\
0 \to & & \tilde{\mathcal{O}} & \to & \mathcal{H}^1_{SD^n}(\tau'^{-1}\mathcal{O}) & \to \alpha_* Q \to 0, & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

the nine lemma gives immediately the

Theorem 1. *There exists a short exact sequence of sheaves:*

$$0 \to \tilde{\mathcal{O}} \to \mathcal{H}^1_{SD^n}(\tau'^{-1}\iota_* \mathcal{O}) \to \alpha_* Q \to 0.$$

Let us then extend the concept of tuboid defined in [1] as follows: If Ω is an open subset of D^n, we shall say that $\Lambda = \bigcup_{x \in \Omega} [\{x\} + i\Lambda_x] \subset D^n + iR^n$ is a *profile with base* Ω if Λ is open and if for any $x \in \Omega$, the fiber Λ_x is an open convex cone in R^n; moreover, an open subset V of Λ will be said to be a *tuboid of profile Λ in $D^n + iR^n$* if, given a compact set $K \subset \Lambda$, there exists $\rho_0 > 0$ such that for every $x + iy \in K$ and every $\rho \in [0, \rho_0]$, the point $x + i\rho y$ belongs to V. We can then extend naturally Bros-Iagolnitzer's result [1] by the

Proposition 2. *Let V be a tuboid of profile Λ in $D^n + iR^n$. There exists then a tuboid $V' \subset V$ of profile Λ in $D^n + iR^n$ such that $V' \cap C^n$ is pseudoconvex.*

Using this proposition together with Dolbeault's resolution of $\mathcal{O}$, we get:

Proposition 3. *The open subsets U of SD^n such that for every $x \in D^n$, $\{t\xi \in R^n : t > 0, (x, \xi) \in U\}$ is convex, are acyclic for the sheaves $\tau'^{-1}[(\iota_* \mathcal{O})|_{D^n}]$ and $(\alpha_* \tau^{-1} \mathcal{O})|_{SD^n}$ hence also for $\tilde{\mathcal{O}}$.*

Adapting the Sato-Malgrange-Fourier (SMF) transform to our particular situation, we define the operator T which transforms sheaves over SD^n in sheaves over S^*D^n by

$$T\mathcal{F} := (R^{n-1}\tau'_{\cdot *})\pi'^{-1}_{\cdot}\mathcal{F}^a,$$

where $\mathcal{F}^a$ denotes indifferently the direct or inverse image of a sheaf $\mathcal{F}$ over SD^n or S^*D^n by the antipodal diffeomorphism $(x, \xi) \to (x, -\xi)$ or $(x, \eta) \to (x, -\eta)$. Let us then define by $\mathcal{C}^\infty$ the sheaf $T\tilde{\mathcal{O}}$. Combining the preceding results with Morimoto's global "Edge of the wedge" theorem and the flabbiness of the sheaves $\mathcal{B}$ and $\mathcal{C}$, we also get:

Proposition 4. *The following hold for every $k \in Z$:*

$$\mathcal{H}^k_{D^n}(\iota_*\mathcal{O}) = \mathcal{H}^n_{D^n}(\iota_*\mathcal{O}).\delta_{k,n};$$
$$(R^k\tau'_{\cdot *})\pi'^{-1}_{\cdot}\tau'^{-1}[(\iota_*\mathcal{O})|_{D^n}] = \pi'^{-1}[(\iota_*\mathcal{O})|_{D^n}].\delta_{k,n-1};$$
$$\mathcal{H}^k_{S^*D^n}(\pi'^{-1}\iota_*\mathcal{O}) = (R^{k-n+1}\tau'_{\cdot *})\pi'^{-1}_{\cdot}\mathcal{H}^1_{SD^n}(\tau'^{-1}\iota_*\mathcal{O}) = \mathcal{H}^n_{S^*D^n}(\pi'^{-1}\iota_*\mathcal{O}).\delta_{k,n};$$
$$(R^k\tau'_{\cdot *})\pi'^{-1}_{\cdot}(\alpha_*\mathcal{Q})^a = \beta_*\mathcal{C}.\delta_{k,n}; \quad (R^k\tau'_{\cdot *})\pi'^{-1}_{\cdot}\tilde{\mathcal{O}}^a = \mathcal{C}^\infty_{\cdot}\delta_{k,n};$$
$$(R^k\tau'_*)\tilde{\mathcal{O}}^a = (R^{k-n+1}\pi'_*)\mathcal{C}^\infty = \pi'_*\mathcal{C}^\infty_{\cdot}\delta_{k,n-1};$$
$$(R^k\tau'_*)\mathcal{H}^1_{SD^n}(\tau'^{-1}\iota_*\mathcal{O}) = (R^{k-n+1}\pi'_*)\mathcal{H}^n_{S^*D^n}(\pi'^{-1}\iota_*\mathcal{O});$$
$$\tau'^{-1}(R^k\tau'_*)\alpha_*\mathcal{Q} = \tau'^{-1}(\pi' \circ \beta)_*\mathcal{C}^a_{\cdot}\delta_{k,n-1}; \quad (R^k\pi'_{\cdot *})\tau'^{-1}_{\cdot}\beta_*\mathcal{C} = \pi'_{\cdot *}\tau'^{-1}_{\cdot}\beta_*\mathcal{C}.\delta_{k,0};$$
$$(R^k\pi'_*)\mathcal{H}^n_{S^*D^n}(\pi'^{-1}\iota_*\mathcal{O}) = \pi'_*\mathcal{H}^n_{S^*D^n}(\pi'^{-1}\iota_*\mathcal{O}).\delta_{k,0};$$
$$(R^k\tau'_*)\mathcal{H}^1_{SD^n}(\tau'^{-1}\iota_*\mathcal{O}) = (R^{n-1}\tau'_*)\mathcal{H}^1_{SD^n}(\tau'^{-1}\iota_*\mathcal{O}).\delta_{k,n-1};$$
$$(R^k\pi'_{\cdot *})\tau'^{-1}_{\cdot}\mathcal{C}^\infty = \pi'_{\cdot *}\tau'^{-1}_{\cdot}\mathcal{C}^\infty_{\cdot}\delta_{k,0};$$
$$(R^k\pi'_{\cdot *})\tau'^{-1}_{\cdot}\mathcal{H}^n_{S^*D^n}(\pi'^{-1}\iota_*\mathcal{O}) = \pi'_{\cdot *}\tau'^{-1}_{\cdot}\mathcal{H}^n_{S^*D^n}(\pi'^{-1}\iota_*\mathcal{O}).\delta_{k,0}.$$

Moreover, there also exist canonical morphisms making exact the following short sequences

$$0 \to \mathcal{C}^\infty \to \mathcal{H}^n_{S^*D^n}(\pi'^{-1}\iota_*\mathcal{O}) \to \beta_*\mathcal{C} \to 0$$
$$0 \to \tau'_*\tau'^{-1}[(\iota_*\mathcal{O})|_{D^n}] \to \iota_*\mathcal{A} \to \pi'_*\mathcal{H}^n_{S^*D^n}(\pi'^{-1}\iota_*\mathcal{O}) \to 0$$
$$0 \to \tilde{\mathcal{O}} \to \tau'^{-1}\pi'_*\mathcal{C}^\infty \to \pi'_{\cdot *}\tau'^{-1}_{\cdot}\mathcal{C}^\infty \to 0$$
$$0 \to \mathcal{H}^1_{SD^n}(\tau'^{-1}\iota_*\mathcal{O})^a \to \tau'^{-1}\pi'_*\mathcal{H}^n_{S^*D^n}(\pi'^{-1}\iota_*\mathcal{O}) \to \pi'_{\cdot *}\tau'^{-1}_{\cdot}\mathcal{H}^n_{S^*D^n}(\pi'^{-1}\iota_*\mathcal{O}) \to 0$$
$$0 \to \pi'_*\mathcal{C}^\infty \to \pi'_*\mathcal{H}^n_{S^*D^n}(\pi'^{-1}\iota_*\mathcal{O})^a \to \iota_*\pi_*\mathcal{C} \to 0$$
$$0 \to \pi'_{\cdot *}\tau'^{-1}_{\cdot}\mathcal{C}^\infty \to \pi'_{\cdot *}\tau'^{-1}_{\cdot}\mathcal{H}^n_{S^*D^n}(\pi'^{-1}\iota_*\mathcal{O})^a \to \pi'_{\cdot *}\tau'^{-1}_{\cdot}\beta_*\mathcal{C} \to 0$$
$$0 \to \tau'^{-1}\pi'_*\mathcal{C}^\infty \to \tau'^{-1}\pi'_*\mathcal{H}^n_{S^*D^n}(\pi'^{-1}\iota_*\mathcal{O})^a \to \tau'^{-1}\pi'_*\beta_*\mathcal{C} \to 0.$$

Combining those results and using SMF transform, we then obtain:

Theorem 2. *There exist canonical morphisms b and sp which make exact the sequence:*

$$0 \longrightarrow (\iota_*\mathcal{O})|_{D^n} \xrightarrow{b} \iota_*\mathcal{A} \xrightarrow{sp} \pi'_*\mathcal{C}^\infty \longrightarrow 0.$$

Pulling this sequence back, we also get

$$0 \longrightarrow \pi'^{-1}[(\iota_*\mathcal{O})|_{D^n}] \xrightarrow{\pi'^{-1}b} \pi'^{-1}\iota_*\mathcal{A} \xrightarrow{\pi'^{-1}sp} \mathcal{C}^\infty \longrightarrow 0.$$

Such exact sequences provide the opportunity to define corresponding wave-front sets for real analytic functions as follows:

Definition. For any open set Ω in D^n and any $f \in \mathcal{A}(\Omega \cap R^n)$, the *wave-front set at infinity* of f may be defined as:

$$W.F.^\infty(f) := \mathrm{supp}_{\mathcal{C}^\infty}(sp\, f)$$

if we consider f as a section of $\pi'^{-1}\iota_*\mathcal{A}$. The meaning of this concept may then be clarified by the following theorems:

Theorem 3. *If U is an open set of SD^n such that, for each $x \in \tau' U$, the set $\{\lambda\xi : \lambda > 0, (x, \xi) \in U\}$ is convex, the space $[(\alpha_* \tau^{-1}\mathcal{O})|_{SD^n}](U)$ is contained in $(\iota_* \mathcal{A})(\tau' U)$ and any $f \in [(\alpha_* \tau^{-1}\mathcal{O})|_{SD^n}](U)$ verifies:*

$$W.F.^\infty(f) \subset \{(x, \eta) \in S^*D^n : x \in \tau' U \text{ and } (x, \xi) \in U \Rightarrow \langle \xi, \eta \rangle \geq 0\}.$$

Theorem 4. *The sheaf $\iota_* \mathcal{A}$ is isomorphic to $(R^{n-1}\tau'_*)[(\alpha_* \tau^{-1}\mathcal{O})|_{SD^n}]$ and this isomorphism can be represented in terms of Čech cohomology as follows: given any open set $\Omega \subset D^n$, any section f of $\iota_* \mathcal{A}$ over Ω and any covering $\mathcal{U}$ of the unit sphere S_{n-1} by sets $\omega_j := \{\xi \in S_{n-1} : \langle \xi, \eta_j \rangle > 0\}$, $(\eta_j \in S^*_{n-1}, j = 1, \cdots, n+1)$, there exist $f_j \in [(\alpha_* \tau^{-1}\mathcal{O})|_{SD^n}]\ (\Omega \times \bigcap_{k \neq j} \omega_k)$ such that $f = \Sigma f_j$; the image of f under the above isomorphism coincides then with the equivalence class of $(f_1, \cdots, f_{n+1})$ in $\check{H}^{n-1}[\mathcal{U}, (\alpha_* \tau^{-1}\mathcal{O})|_{SD^n}]$.*

Theorem 5. *For any point (x, η) of S^*D^n, one has $(x, \eta) \notin W.F.^\infty(f)$ if and only if there exist some neighborhood Ω of x, some open convex cones $\Gamma_j \subset \{\xi : \langle \xi, \eta \rangle < 0\}$ $(j = 1, \cdots, J)$ and some $f_j \in [(\alpha_* \tau^{-1}\mathcal{O})|_{SD^n}]$ $(\Omega \times (\Gamma_j \cap S_{n-1}))$ such that $f = \Sigma f_j$.*

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