79. A New Formulation of Local Boundary Value Problem in the Framework of Hyperfunctions. I

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Komatsu-Kawai [7] and Schapira [9] formulated non-characteristic boundary value problem in the framework of hyperfunctions for single linear partial differential equations. They defined the boundary values of hyperfunction solutions and proved the uniqueness of solutions using the dual version of the Cauchy-Kovalevskaja theorem. Later Kataoka [6] introduced the notion of mild hyperfunctions and clarified the meaning of boundary values explicitly using defining functions. The notion of mild hyperfunctions has been extended to that of F-mild hyperfunctions by Ôaku [8] by more elementary method. On the other hand, Kashiwara-Kawai [3] formulated non-characteristic boundary value problem for elliptic systems of partial differential equations by using the sheaf $\mathcal{C}_{N|X}$.

Here we propose a new method to formulate non-characteristic boundary value problem for general systems of partial differential equations by using the sheaf of hyperfunctions with holomorphic parameters. This is closely connected with the basic idea that hyperfunctions are sums of boundary values of holomorphic functions. Hence it enables us to reduce the study of boundary value problem to the study of holomorphic solutions of the equations in the complex domain. We shall study the relation between this formulation and F-mild hyperfunctions in the next paper.

Since we are interested in the local properties, we work only in Euclidean spaces. Put $M=\boldsymbol{R}^n \ni x=(x_1, x')$ with $x'=(x_2, \cdots, x_n)$, $X=\boldsymbol{C}^n \ni z=(z_1, z')$ with $z'=(z_2, \cdots, z_n)$. We regard M and X as products $M=\boldsymbol{R}\times N$ and $X=\boldsymbol{C}\times Y$ respectively with $N=\boldsymbol{R}^{n-1}$ and $Y=\boldsymbol{C}^{n-1}$. For the sake of simplicity of the notation, we identify $\{0\}\times N$ and $\{0\}\times Y$ with N and Y respectively. Set $\tilde{M}=\boldsymbol{R}\times Y$, $M_+=\boldsymbol{R}_+\times N$, $\tilde{M}_+=\boldsymbol{R}_+\times Y$ with $\boldsymbol{R}_+=\{x_1\in \boldsymbol{R}\,;\, x_1\geqq 0\}$. Let

$$\iota : \operatorname{int} M_+ \longrightarrow M, \qquad \tilde{\iota} : \operatorname{int} \tilde{M}_+ \longrightarrow \tilde{M}$$

be the natural embeddings, where int denotes the interior of a set. Set $\mathcal{B}_{N|M+}=(\iota_*\iota^{-1}\mathcal{B}_M)|_N$, where $\mathcal{B}_M$ denotes the sheaf on M of hyperfunctions. The sheaf $\mathcal{B}_{N|M+}$ is, roughly speaking, the sheaf of hyperfunctions defined on the positive side of N. By the flabbiness of $\mathcal{B}_M$, we have an isomorphism

$$\mathcal{B}_{N|M_+} \cong (\Gamma_{M_+}(\mathcal{B}_M)/\Gamma_N(\mathcal{B}_M))|_N.$$

We begin by expressing $\mathcal{B}_{N|M_+}$ as a cohomology group with coefficients in the sheaf of hyperfunctions with holomorphic parameters. For this purpose let $\mathcal{BO}_{\tilde{M}}$ be the sheaf on $\tilde{M}$ of hyperfunctions with holomorphic parameters $z'=(z_2, \cdots, z_n)$. Then we have

$$\Gamma_{M_+}(\mathcal{B}_M) \cong \mathcal{H}^{n-1}_M(\Gamma_{\tilde{M}_+}(\mathcal{BO}_{\tilde{M}})), \qquad \Gamma_N(\mathcal{B}_M) \cong \mathcal{H}^{n-1}_M(\Gamma_Y(\mathcal{BO}_{\tilde{M}})).$$

From this we get

$$\mathcal{B}_{N|M_+} \cong \mathcal{H}^{n-1}_M(\iota_*\iota^{-1}\mathcal{BO}_{\tilde{M}})|_N$$

since M is purely $(n-1)$-codimensional with respect to $\iota_*\iota^{-1}\mathcal{BO}_{\tilde{M}}$ and there is an exact sequence

$$0 \longrightarrow \Gamma_Y(\mathcal{BO}_{\tilde{M}}) \longrightarrow \Gamma_{\tilde{M}_+}(\mathcal{BO}_{\tilde{M}}) \longrightarrow \iota_*\iota^{-1}\mathcal{BO}_{\tilde{M}} \longrightarrow 0.$$

Now we introduce a sheaf $\tilde{\mathcal{B}}_{N|M_+}$ as follows.

Definition. $\tilde{\mathcal{B}}_{N|M_+} = \mathcal{H}^{n-1}_N(\mathcal{BO}_{Y|\tilde{M}_+})$ with $\mathcal{BO}_{Y|\tilde{M}_+} = (\iota_*\iota^{-1}\mathcal{BO}_{\tilde{M}})|_Y$.

By virtue of an abstract edge of the wedge theorem due to Kashiwara-Laurent (Théorème 1.4.1 of [4]), we have the following vanishing theorem.

Proposition 1. $\mathcal{H}^j_N(\mathcal{BO}_{Y|\tilde{M}_+})=0$ *if* $j \neq n-1$.

The following proposition follows from Proposition 1 and the fact that the flabby dimension of $\mathcal{BO}_{Y|\tilde{M}_+}$ is $n-1$.

Proposition 2. $\tilde{\mathcal{B}}_{N|M_+}$ *is a flabby sheaf on* N.

Let q and $\tilde{q}$ be the natural embeddings

$$q: N=\{0\}\times \mathbf{R}^{n-1} \longrightarrow M, \qquad \tilde{q}: Y=\{0\}\times \mathbf{C}^{n-1} \longrightarrow \tilde{M}.$$

Then natural homomorphisms

$$\Gamma_{M_+}(\mathcal{B}_M)|_N \cong q^{-1}\mathcal{H}^{n-1}_M(\Gamma_{\tilde{M}_+}(\mathcal{BO}_{\tilde{M}})) \longrightarrow \mathcal{H}^{n-1}_N(\tilde{q}^{-1}\Gamma_{\tilde{M}_+}(\mathcal{BO}_{\tilde{M}})),$$
$$\Gamma_N(\mathcal{B}_M)|_N \longrightarrow \mathcal{H}^{n-1}_N(\tilde{q}^{-1}\Gamma_Y(\mathcal{BO}_{\tilde{M}}))$$

are induced and the last homomorphism is an isomorphism. Thus we get a natural homomorphism

$$\alpha: \mathcal{B}_{N|M_+} \longrightarrow \tilde{\mathcal{B}}_{N|M_+}$$

of $\mathcal{D}_X|_Y$-modules ($\mathcal{D}_X$ denotes the sheaf of rings of partial differential operators with holomorphic coefficients on X).

Theorem 1. *The homomorphism* α *is injective.*

This theorem can be proved by the curvilinear wave expansion (Radon transformation) applied to hyperfunctions with holomorphic parameters.

Theorem 2. *Let* $\mathcal{M}$ *be a coherent* $\mathcal{D}_X$*-module* (*i.e. a system of partial differential equations*) *defined in a neighborhood of a point of* N. *Suppose* Y *is non-characteristic with respect to* $\mathcal{M}$. *Then there exists a natural isomorphism*

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M_+}) \cong \mathcal{E}xt^j_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N)$$

for any $j \in \mathbf{Z}$; *here* $\mathcal{M}_Y$ *denotes the tangential system of* $\mathcal{M}$ (cf. Kashiwara [2]).

Proof. We have a triangle (see Hartshorne [1] for the notion of derived categories and triangles)

$$
\begin{tikzcd}
& \boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}) \arrow[dl, "+1"'] & \\
\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\tilde{M}_-}(\mathcal{B}\mathcal{O}_{\tilde{M}})|_Y) \arrow[rr] & & \boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}_{\tilde{M}}|_Y), \arrow[ul]
\end{tikzcd}
$$

where $\tilde{M}_-=\{x_1 \in \boldsymbol{R}\,;\, x_1 \leqq 0\} \times Y$, and $\boldsymbol{R}\,\mathcal{H}om$ denotes the right derived functor of $\mathcal{H}om$. Since Y is non-characteristic with respect to $\mathcal{M}$, we get

$$\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \boldsymbol{R}\Gamma_{\tilde{M}_-}(\mathcal{B}\mathcal{O}_{\tilde{M}})|_Y)=0$$

by virtue of Theorem 2.2.1 of Kashiwara-Schapira [5], and hence an isomorphism

$$\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}_{\tilde{M}}|_Y) \cong \boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}).$$

On the other hand, we have

$$\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}_{\tilde{M}}|_Y) \cong \boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X|_Y) \cong \boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)$$

by virtue of the Cauchy-Kovalevskaja theorem due to Kashiwara [2]; here $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on X. Combining these isomorphisms, we get an isomorphism

$$\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}) \cong \boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y).$$

Thus we have

$$
\begin{aligned}
\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M_+}) &= \boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \boldsymbol{R}\Gamma_N(\mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}))[n-1] \\
&= \boldsymbol{R}\Gamma_N(\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}))[n-1] \\
&\cong \boldsymbol{R}\Gamma_N(\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y))[n-1] \\
&= \boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N).
\end{aligned}
$$

This completes the proof.

Corollary. *Let $\mathcal{M}$ be as in Theorem 2. Then there exists a natural homomorphism*

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+}) \longrightarrow \mathcal{E}xt^j_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N)$$

for any $j \in \boldsymbol{Z}$. In particular, the homomorphism

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+}) \longrightarrow \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N)$$

is injective.

Thus we have defined the homomorphism of boundary values and at the same time proved the uniqueness of solutions (a generalization of Holmgren's uniqueness theorem).

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