78. *The L^p-boundedness of Pseudo-differential Operators Satisfying Estimates of Parabolic Type and Product Type*

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(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 12, 1984)

In this paper we consider symbols $P(x,\xi)$ satisfying certain estimates such as $|\partial_{\xi_l}^k P(x,\xi)|\leqq C(1+\xi_l^2)^{-k/2}$ for every $l=1,2,\cdots,n$ and $k=0,1,\cdots,n+1$, and we give a sufficient condition under which the associated pseudo-differential operators $P(x,D_x)$ are bounded on $L^p=L^p(\boldsymbol{R}^n)$, where $1<p<\infty$.

We shall also show that our condition is sharp, by constructing an operator which is not L^p-bounded for any $1<p<\infty$.

To obtain the result we establish a version of the Littlewood-Paley decomposition theorem of the space $L^p(\boldsymbol{R}^n)$ of parabolic type and product type.

1. Statement of the theorem. Let $n_1, n_2, \cdots, n_N$ be a family of positive integers. We put $n=n_1+n_2+\cdots+n_N$ and

$$\Lambda_\nu=\{l\in \boldsymbol{N}\,;\, n_1+\cdots+n_{\nu-1}+1\leqq l\leqq n_1+\cdots+n_{\nu-1}+n_\nu\}$$

for $\nu=1,2,\cdots,N$.

We regard $\boldsymbol{R}^n$ as $\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}\times\cdots\times\boldsymbol{R}^{n_N}$, and denote $x\in\boldsymbol{R}^n$ as $x=(x^{(1)},\cdots,x^{(N)})$, where $x^{(\nu)}=(x_l)_{l\in\Lambda_\nu}\in\boldsymbol{R}^{n_\nu}$. We also give a weight $M=(M^{(1)},\cdots,M^{(N)})$ to $\boldsymbol{R}^n$, where each $M^{(\nu)}=(m_l)_{l\in\Lambda_\nu}$ satisfies $\min_{l\in\Lambda_\nu} m_l=1$.

For $y=(y_l)_{l\in\Lambda_\nu}\in\boldsymbol{R}^{n_\nu}$ we define the action of $t\in\boldsymbol{R}^+=\{t\,;\,t\geqq 0\}$ to y by $t^{M^{(\nu)}}y=(t^{m_l}y_l)_{l\in\Lambda_\nu}$, and we denote by $[y]_\nu$ the only positive number t satisfying $t^{-M^{(\nu)}}y=(t^{-1})^{M^{(\nu)}}y\in\{y\in\boldsymbol{R}^{n_\nu}\,;\,|y|=1\}$. (For $y=0$ we set $[0]_\nu=0$.) For $x\in\boldsymbol{R}^n$ we put $t^Mx=(t^{M^{(1)}}x^{(1)},\cdots,t^{M^{(N)}}x^{(N)})$. If $f(x)$ is a function on $\boldsymbol{R}^n$, then for $\nu=1,2,\cdots,N$ and $y\in\boldsymbol{R}^{n_\nu}$ we write

$$\Delta_y^{(\nu)}f(x)=f(x^{(1)},\cdots,x^{(\nu)}-y,\cdots,x^{(N)})-f(x).$$

Now we introduce a notion to state our main theorem.

Definition. We call a set of functions $\{\omega_1(t_1),\omega_2(t_1,t_2),\cdots,\omega_N(t_1,t_2,\cdots,t_N)\}$ a *modulus of continuity* if it satisfies the following three conditions:

1) Each $\omega_\nu(t_1,t_2,\cdots,t_\nu)$ is a function on $(\boldsymbol{R}^+)^\nu$ into $\boldsymbol{R}^+$.

2) $\omega_\nu(t_1,t_2,\cdots,t_\nu)$ is monotone-increasing and concave for each t_k, where $1\leqq k\leqq\nu$.

3) $\omega_{\nu+\mu}(t_1,t_2,\cdots,t_{\nu+\mu})\leqq\min\{2^\mu\omega_\nu(t_1,\cdots,t_\nu),\,2^\nu\omega_\mu(t_{\nu+1},\cdots,t_{\nu+\mu})\}$.

Theorem. *The following three conditions concerning moduli of* *continuity are equivalent:*

1) $\displaystyle\int_0^1 \cdots \int_0^1 \frac{\omega_\nu(t_1, t_2, \cdots, t_\nu)^2}{t_1 t_2 \cdots t_\nu} dt_1 dt_2 \cdots dt_\nu < \infty$

for every $\nu = 1, 2, \cdots, N$.

2) *Suppose that a symbol* $P(x, \xi)$ *satisfies the following estimates* $(*\mu)$ *for all* $\mu = 0, 1, \cdots, N$:

(*0) *For every* $\nu = 1, 2, \cdots, N$, $l \in \Lambda_\nu$ *and* $k = 0, 1, \cdots, n+1$ *we have*

$$|\partial_{\xi_l}^k P(x, \xi)| \leqq C(1 + [\xi^{(\nu)}]_\nu)^{-m_l k}.$$

$(*\mu)$ *For every* $\nu = 1, 2, \cdots, N$, $1 \leqq \nu(1) < \nu(2) < \cdots < \nu(\mu) \leqq N$, $y_1 \in \boldsymbol{R}^{n_{\nu(1)}}, \cdots, y_\mu \in \boldsymbol{R}^{n_{\nu(\mu)}}$, $l \in \Lambda_\nu$ *and* $k = 0, 1, \cdots, n+1$ *we have*

$$|\Delta_{y_1}^{(\nu(1))}(\cdots(\Delta_{y_\mu}^{(\nu(\mu))}\{\partial_{\xi_l}^k P(x, \xi)\})\cdots)| \leqq C\omega_\mu([y_1]_{\nu(1)}, \cdots, [y_\mu]_{\nu(\mu)})(1 + [\xi^{(\nu)}]_\nu)^{-m_l k}.$$

Then the associated pseudo-differential operator $P(x, D_x)$ *is bounded on* $L^p(\boldsymbol{R}^n)$ *for all* $1 < p < \infty$.

3) *For every symbol* $P(x, \xi)$ *satisfying the estimates* $(*\mu)$ *for all* $\mu = 0, 1, \cdots, N$ *there exists* $1 < p < \infty$ *such that* $P(x, D_x)$ *is bounded on* $L^p(\boldsymbol{R}^n)$.

2. Outline of the proof of 2)→3)→1) and remarks. The assertion 2)→3) is trivial. If a modulus of continuity $\{\omega_1(t_1), \cdots, \omega_N(t_1, t_2, \cdots, t_N)\}$ does not satisfy the condition 1), then we can construct a symbol $P(x, \xi)$ such that the estimate $(*\mu)$ holds for every $\mu = 0, 1, \cdots, N$ and that the associated operator $P(x, D_x)$ is not bounded on $L^p(\boldsymbol{R}^n)$ for any $1 < p < \infty$. This implies the assertion 3)→1).

Remark 1. It was shown in Coifman-Meyer [1] that the condition $\int_0^1 t^{-1}\omega_1(t)^2 dt < \infty$ is necessary for the L^p-boundedness of the operators associated with symbols satisfying (*0) and (*1). In case $N \geqq 2$, the hypothesis 1) is satisfied if

$$(***) \qquad \int_0^1 t^{-1}(-\log t)^{N-1}\omega_1(t)^2 dt < \infty,$$

since we have

$$\int_0^1 \cdots \int_0^1 \frac{\omega_\nu(t_1, \cdots, t_\nu)^2}{t_1 \cdots t_\nu} dt_1 \cdots dt_\nu \leqq \int_0^1 \cdots \int_0^1 \{2^{\nu-1} \min_l \omega_1(t_l)\}^2 \frac{dt_1 \cdots dt_\nu}{t_1 \cdots t_\nu} = \nu 4^{\nu-1} \int_0^1 (-\log t)^{\nu-1} \frac{\omega_1(t)^2}{t} dt.$$

On the other hand, if $\omega_1(t)$ is a continuous, monotone-increasing, concave function which does not satisfy $(***)$, then we can construct a modulus of continuity which does not satisfy the condition 1) by putting $\omega_\nu(t_1, \cdots, t_\nu) = 2^{\nu-1}\omega_1(\min\{t_1, \cdots, t_\nu\})$.

Remark 2. The L^p-boundedness of pseudo-differential operators with symbols satisfying similar estimates as $(*\mu)$ was shown in [1] in the case $N=1$ and $M=(1, \cdots, 1)$. They assumed the estimates of the derivative $\partial_\xi^\alpha P$ for each $\alpha \in N^n$. Muramatu-Nagase [3] showed that the estimates of $\partial_\xi^\alpha P$ for α satisfying $|\alpha| \leqq n+2$ are sufficient. The case $N=1$ and $M \neq (1, \cdots, 1)$ was treated by Yamazaki [6].

For other L^p-boundedness theorems of this type, see Mossaheb-Okada [2] and Nagase [4].

3. Outline of the proof of 1)→2). Let $\psi_0(t)$ be a C^∞-function on R^+ satisfying $0 \leqq \psi_0(t) \leqq 1$, $\psi_0(t)=1$ $(t \leqq 1)$ and $\psi_0(t)=0$ $(t \geqq 4/3)$. For $j=1, 2, \cdots$ we put $\psi_j(t)=\psi_0(2^{-j}t)-\psi_0(2^{1-j}t)$. Then we have the following.

Lemma 1. *Suppose that $a \in R^n$, $1 \leqq \nu \leqq N$, $K=(k_1, \cdots, k_\nu) \in N^\nu$ and $u(x) \in L^p(R^n)$, where $1<p<\infty$. If we put*

$$u_{a,K}(x)=\mathcal{F}^{-1}[\exp(i \textstyle\sum_{j=1}^{\nu} a^{(j)} \cdot 2^{-k_j M^{(j)}} \xi^{(j)}) \psi_{k_1}([\xi^{(1)}]_1) \cdots \psi_{k_\nu}([\xi^{(\nu)}]_\nu) \hat{u}(\xi)](x).$$

Then we have the estimate

$$\|(\textstyle\sum_{K \in N^\nu} |u_{a,K}(x)|^2)^{1/2}\|_{L^p} \leqq B(\prod_{j=1}^{\nu} \log(2+[a^{(j)}]_j)) \|u\|_{L^p}$$

for some constant B independent of a.

This lemma can be shown by virtue of a non-isotropic version of the Calderón-Zygmund decomposition theorem (see Stein [5], Chap. I) and the use of the Rademacher functions (see [5], Chap. IV). Lemma 1 and the standard duality argument yield the following.

Lemma 2. *Suppose that $1<p<\infty$, $1 \leqq \nu \leqq N$ and $B>1$. For $K \in N^\nu$ we denote by I_K the set of $\xi \in R^n$ satisfying $[\xi^{(j)}]_j<B$ for all j such that $1 \leqq j \leqq \nu$ and $k_j=0$ and $2^{k_j}B^{-1}<[\xi^{(j)}]_j<2^{k_j}B$ for all j such that $1 \leqq j \leqq \nu$ and $k_j \geqq 1$. If a family of functions $\{u_K(x)\}_{K \in N^\nu}$ satisfies the condition* supp $u_K(\xi) \subset I_K$ *and the estimate*

$$\|(\textstyle\sum_{K \in N^\nu} |u_K(x)|^2)^{1/2}\|_{L^p(R^n)}<\infty,$$

then the sum $u(x)=\sum_{K \in N^\nu} u_K(x)$ is well-defined in $L^p(R^n)$, and we have the estimate $\|u(x)\|_{L^p} \leqq C \|(\sum_K |u_K(x)|^2)^{1/2}\|_{L^p}$.

The theorem can be proved in the same manner as in [1]. We decompose "reduced symbols" into 2^N parts, and estimate each part by virtue of Lemma 1 and Lemma 2. Details will be published elsewhere.

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