74. *Dualizing with respect to s-tuples*

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1. Introduction. In a projective plane, if the roles of lines and points are interchanged, the dual geometry is obtained. Similar concept was introduced in a field of the design of experiments by Bose and Nair [1], who derived a new class of designs, by interchanging blocks and treatments in a given class of block designs. This concept of interchanging the roles of blocks and treatments is usually named as "DUALIZATION". We denote the dual of the design D by D_1^*. This dualization, that is, writing the block numbers of blocks in which a treatment occurs in the original design, is extended to another concept as writing the block numbers of blocks in which a pair of treatments occurs in the original design. This is named as "Dualization with respect to (w.r.t.) pairs", which is denoted by D_2^* for a given block design D, and is dealt with in Mohan and Kageyama [6]. In this note, the concept of dualization w.r.t. pairs is further generalized in the form as "Dualization w.r.t. s-tuples" for $s \geqq 1$. This dual design is denoted by D_s^*. Applying this technique to certain designs yields new block designs D_s^* for some values of s. For the description of some technical terms in designs, we refer the reader to Raghavarao [7].

2. Method. We here consider an equireplicated and equiblock-sized design in which the number of treatments (with the replication number r) is v and the number of blocks (of size k) is b. The present method is as follows: Number the blocks of a given block design D. Now in D_s^* if the i-th block of D includes an s-tuple, then the corresponding block of D_s^* will have the i-th treatment of D_s^*. This D_s^* coincides with the known cases described in the introduction when $s=1$ and 2.

For a given block design D with parameters v, b, r and k, it is obvious that its dual design D_s^* w.r.t. s-tuples, for $s<k$, is characterized by the parameters in the following form:

$v'=b,\ b'=\binom{v}{s},\ r'=\binom{k}{s},$

$k'=$the number of times s-tuples of treatments occur in the original design,

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$$\lambda' = \binom{\text{the number of treatments common to any two blocks}}{s}.$$

Note that if the number of times s-tuples of treatments occur and the number of treatments common to any two blocks in the original design are not constant, then the values of k' and λ' are also varying.

We will deal mainly with a case of $s \geqq 3$ to give a new insight of this concept, because a case of $s=1$ is well-known as the usual dualization and there is some discussion in Mohan and Kageyama [6] when $s=2$. In particular, constructions of partially balanced incomplete block (PBIB) designs are here discussed a little to show some advantage of the present approach.

3. Statement. There are various kinds of block designs. Among them, we shall utilize a t-design as a starting block design. Properties of t-designs are discussed in many literature (cf. Hedayat and Kageyama [3], Kageyama and Hedayat [4], as survey papers). It is well known that for each $0 \leqq s \leqq t$ every t-(v, k, λ_t) design is an s-(v, k, λ_s) design with $\lambda_s = \lambda_t \binom{v-s}{t-s} \Big/ \binom{k-s}{t-s}$. Usually, we put $b=\lambda_0$ and $r=\lambda_1$ for symmetry of notation. As one of t-designs having a property suitable to our purpose, we have a class of quasi-symmetric t-designs. A t-design is said to be quasi-symmetric if any of its two blocks have either x or y common treatments, where $x \neq y$ (i.e., the number of treatments incident with two blocks takes just two distinct values).

Theorem. *Dualizing a quasi-symmetric t-(v, k, λ_t) design with the above x and y w.r.t. s-tuples yields, for $1 \leqq s \leqq t$, a 2-associate* PBIB *design D_s^* with parameters $v'=b$, $b'=\binom{v}{s}$, $r'=\binom{k}{s}$, $k'=\lambda_s$, $\lambda_1'=\binom{x}{s}$, $\lambda_2'=\binom{y}{s}$; $n_1=[k(r-1)-y(b-1)]/(x-y)$, $n_2=[x(b-1)-k(r-1)]/(x-y)$, where $\binom{c}{d}=0$ if $c<d$.*

Proof. The values of v', b', r', k', λ_1', λ_2', are obvious from the definition of the dualization w.r.t. s-tuples. The values of n_1 and n_2 follow from Corollary 2.1 of Shah [9]. Thus, the proof is completed.

Remark. If we let $x<y$, then $k(r-1)>(b-1)x$ and $k(r-1)<(b-1)y$, which show that n_1 and n_2 in the theorem are positive. If the design is a linked block type (i.e., $x=y$), then $n_1=n_2=0$.

4. Some discussions. In the theorem, if $x=0$, it follows (cf. Shah [9]) that $y=(r-\lambda_2-k+k\lambda_2)/(r-1)$ (=an integer), $n_1=b-1-k(r-1)^2/(r-\lambda_2-k+k\lambda_2)$. Thus, if $x=0$, $v=nk$ for some integer n, and $b=v+r-1$, then we obtain $n_1=n-1$ and $y=k^2/v$. This observation implies that we can utilize an affine resolvable t-design as a starting quasi-symmetric block design in the theorem. In fact, an affine resolvable design has $x=0$. Available affine resolvable t-designs yield some new PBIB designs. For example, Kimberley [5] showed that all resolvable 3-(v, k, λ_3) designs are affine resolvable if and only if they are 3-$(4\lambda_3+4, 2\lambda_3+2, \lambda_3)$ designs. In this case, they have other parameters as $b=2(4\lambda_3+3)$, $r=4\lambda_3+3$, $\lambda_2=2\lambda_3+1$, $x=0$ and $y=1+\lambda_3$. Hence the theorem yields three group divisible 2-associate PBIB designs (D_1^*, D_2^* and D_3^*) with parameters $v'=2(4\lambda_3+3)$ (the treatments consisting of $4\lambda_3+3$ groups of 2 treatments each), $b'=\binom{4\lambda_3+4}{s}$, $r'=\binom{2\lambda_3+2}{s}$, $k'=\lambda_s$, $\lambda_1'=0$, $\lambda_2'=\binom{\lambda_3+1}{s}$; $n_1=1$, $n_2=4(2\lambda_3+1)$ for $s=1, 2, 3$. This is a new series of group divisible PBIB designs (semi-regular type for $s=1$, and regular type for $s=2, 3$). As another illustration with $x>0$ and $y>0$ for the theorem, we can take an affine μ-resolvable t-design for $\mu\geqq 2$, $t=2$ and 3, as a starting design. In this case, note that $x=k+\lambda_2-r$ and $y=k^2/v$, which yield a 2-associate PBIB design.

If a t-design has m distinct block intersection numbers, our technique may produce PBIB designs D_s^* with *at least m* associate classes for each of integers s satisfying $1\leqq s\leqq t$. For example, it is known (cf. Cameron [2], Ray-Chaudhuri and Wilson [8]) that in a nontrivial t-design there are at least α (or $\alpha+1$) distinct block intersection numbers for $t=2\alpha$ (or $2\alpha+1$), respectively, and in particular there are exactly α distinct block intersection numbers if and only if the 2α-design is tight. So, a tight 2α-design yields a PBIB design with at least α associate classes. As an illustration, there exists a tight 4-(23, 7, 1) design with $x=1$ and $y=3$, which yields *three* 2-associate PBIB designs. Incidentally, a nontrivial t-design may produce a PBIB design having at least α (or $\alpha+1$) associate classes for $t=2\alpha$ (or $2\alpha+1$), respectively.

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