8. *Exponential Quadratic Splines*

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1. Introduction and consistency relations. In practical applications, curves and surfaces which appear as smooth as possible to the human viewer are to be fitted through given points in a plane or in space, so it has been common practice to restrict attention to splines which are piecewise polynomials with continuous first or second derivatives. However, certain generalized splines are sometimes more useful as they permit the variation of additional parameters. In this regard, various results have been already obtained on exponential cubic splines and the other generalized cubic splines (Späth [4]).

In the present note, we consider exponential quadratic splines and their consistency relations among function or derivative values at mesh and mid points.

Let Δ_n be a partition of the interval [0, 1] with the following mesh points:

(1.1) $$\Delta_n: 0=x_0<x_1<\cdots<x_n=1,$$

and $\Lambda=(\lambda_0, \lambda_1, \cdots, \lambda_{n-1})$ be a system of n non-negative numbers. Then we define an exponential quadratic spline $s(x)\in C^1[0,1]$ associated to (Δ_n, Λ) as follows:

(1.2) $$s(x)=\alpha_i+\beta_i\phi_i\{(x-x_i)/h_i\}+\gamma_i\psi_i\{(x-x_i)/h_i\}$$
$$x_i\leq x\leq x_{i+1},\quad h_i=x_{i+1}-x_i,\quad i=0,1,\cdots,n-1$$

where for $\lambda_i>0$

(1.3) (i) $\phi_i(x)=(1/\lambda_i)\sinh(\lambda_i x)$
(ii) $\psi_i(x)=(2/\lambda_i^2)\{\cosh(\lambda_i x)-1\}$ ([5]),

and for $\lambda_i=0$, $\varphi_i(x)=x$ and $\psi_i(x)=x^2$.

By simple calculation, we have

(1.4) $$\phi_i(x)\longrightarrow x \quad\text{and}\quad \psi_i(x)\longrightarrow x^2 \quad\text{as}\quad \lambda_i\longrightarrow 0.$$

Since $s(x)$ depends upon six parameters $\alpha_j, \beta_j, \gamma_j$, $j=i, i+1$ on $[x_i, x_{i+2}]$ and continuity conditions of $s^{(k)}(x)$, $k=0,1$ at $x=x_{i+1}$ gives us two conditions toward the determination of these parameters, five quantities s_j, $j=i, i+1, i+2$ and $s_{j+1/2}$, $j=i, i+1$ must be interrelated, i.e., we have

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Theorem 1. *For $i=0, 1, \cdots, n-1$, we have*

$$\frac{\lambda_i}{h_i}\cdot\frac{s_i}{2\sinh(\lambda_i/2)}+Cs_{i+1}+\frac{\lambda_{i+1}}{h_{i+1}}\cdot\frac{s_{i+2}}{2\sinh(\lambda_{i+1}/2)} \tag{1.5}$$
$$=\frac{\lambda_i}{h_i}\cdot\frac{s_{i+1/2}}{\tanh(\lambda_i/4)}+\frac{\lambda_{i+1}}{h_{i+1}}\cdot\frac{s_{i+3/2}}{\tanh(\lambda_{i+1}/4)}$$

where $s_i=s(x_i)$, $s_{i+1/2}=s((x_i+x_{i+1})/2)$ and C is determined by substituting $s(x)\equiv 1$ into the above relation.

Remark. The above consistency relation leads to the well-known one for quadratic spline as $\lambda_i, \lambda_{i+1}\to 0$:

$$\frac{s_i}{h_i}+3\left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)s_{i+1}+\frac{s_{i+2}}{h_{i+1}} \tag{1.6}$$
$$=4\left(\frac{s_{i+1/2}}{h_i}+\frac{s_{i+3/2}}{h_{i+1}}\right) \qquad ([2]).$$

Proof. Here we shall prove the relation for $i=0$. Since

$$s(x)=\alpha_0+\beta_0\phi_0(x/h_0)+\gamma_0\psi_0(x/h_0), \qquad x_0\le x\le x_1, \tag{1.7}$$

we have

$$s_{1/2}=\alpha_0+\beta_0\phi_0(1/2)+\gamma_0\psi_0(1/2) \tag{1.8}$$
$$h_0s'_{1/2}=\beta_0\phi'_0(1/2)+\gamma_0\psi'_0(1/2)$$

from which follows

$$s(x)=s_{1/2}+\{\phi_0(x/h_0)-\phi_0(1/2)\}\frac{h_0s'_{1/2}}{\phi'_0(1/2)} \tag{1.9}$$
$$+\left[\psi_0(x/h_0)-\psi_0(1/2)-\frac{\psi'_0(1/2)}{\phi'_0(1/2)}\{\phi_0(x/h_0)-\phi_0(1/2)\}\right]\gamma_0$$
$$x_0\le x\le x_1.$$

Hence we have a system of equations with respect to $s'_{1/2}$ and γ_0:

$$s_0=s_{1/2}-\frac{h_0}{\lambda_0}\tanh(\lambda_0/2)s'_{1/2}+\frac{2\gamma_0}{\lambda_0^2}\{1-1/\cosh(\lambda_0/2)\} \tag{1.10}$$
$$s_1=s_{1/2}+\frac{h_0}{\lambda_0}\{2\sinh(\lambda_0/2)-\tanh(\lambda_0/2)\}s'_{1/2}$$
$$+\frac{2\gamma_0}{\lambda_0^2}\{1-1/\cosh(\lambda_0/2)\}.$$

Since $\lambda_0>0$, i.e., $\sinh(\lambda_0/2)\ne 0$ and $\cosh(\lambda_0/2)>1$, we have

$$s'_{1/2}=\frac{\lambda_0(s_1-s_0)}{2h_0\sinh(\lambda_0/2)} \tag{1.11}$$
$$\gamma_0=\frac{\lambda_0^2}{2}\{1-1/\cosh(\lambda_0/2)\}^{-1}\left\{s_0-s_{1/2}+\frac{s_1-s_0}{2\cosh(\lambda_0/2)}\right\}.$$

Similarly we have

$$s'_{3/2}=\frac{\lambda_1(s_2-s_1)}{2h_1\sinh(\lambda_1/2)} \tag{1.12}$$
$$\gamma_1=\frac{\lambda_1^2}{2}\{1-1/\cosh(\lambda_1/2)\}^{-1}\left\{s_1-s_{3/2}+\frac{s_2-s_1}{2\cosh(\lambda_1/2)}\right\}.$$

Using again (1.9), we have

$$(1.13)\qquad \begin{aligned} s_0' &= \frac{1}{\cosh(\lambda_0/2)} s_{1/2}' - \frac{2\gamma_0}{\lambda_0 h_0}\tanh(\lambda_0/2) \\ s_1' &= \frac{\cosh(\lambda_0)}{\cosh(\lambda_0/2)} s_{1/2}' + \frac{2\gamma_0}{\lambda_0 h_0}\tanh(\lambda_0/2). \end{aligned}$$

Hence, continuity condition of s' at $x=x_1$ yields

$$(1.14)\qquad \begin{aligned} &\frac{1}{\cosh(\lambda_1/2)} s_{3/2}' - \frac{2\gamma_1}{\lambda_1 h_1}\tanh(\lambda_1/2) \\ &\quad = \frac{\cosh(\lambda_0)}{\cosh(\lambda_0/2)} s_{1/2}' + \frac{2\gamma_0}{\lambda_0 h_0}\tanh(\lambda_0/2). \end{aligned}$$

By substituting (1.11) and (1.12) into (1.14), we have the desired consistency relation.

Similarly we have

Theorem 2. *For $i=0, 1, \cdots, n-1$, we have*

$$(1.15)\qquad \begin{aligned} &\frac{\lambda_{i+1}(s_{i+2}-s_{i+1})}{2h_{i+1}\tanh(\lambda_{i+1}/2)} - \frac{\lambda_i(s_{i+1}-s_i)}{2h_i\tanh(\lambda_i/2)} \\ &\quad = \frac{h_{i+1}}{\lambda_{i+1}}\sinh(\lambda_{i+1}/2)s_{i+3/2}'' + \frac{h_i}{\lambda_i}\sinh(\lambda_i/2)s_{i+1/2}''. \end{aligned}$$

Remark. The above consistency relation leads to the well-known one for quadratic spline as $\lambda_i, \lambda_{i+1} \to 0$:

(1.16) $(s_{i+2}-s_{i+1})/h_{i+1}-(s_{i+1}-s_i)/h_i=(1/2)(h_{i+1}s_{i+3/2}''+h_i s_{i+1/2}'')$ ([2]).

2. Example. In this section we consider an application of the above stated consistency relations to approximation of function $f(x)=1-\exp(-100x)$. Here we consider quadratic spline interpolation at mid-point so that

$$(2.1)\qquad \begin{cases} s_{i+1/2}=f_{i+1/2}, & i=0, 1, \cdots, 9 \\ s_0=f_0, & s_{10}=f_{10} \\ h_i=h(=0.1), & i=0, 1, \cdots, 9. \end{cases}$$

In the following table, we choose $\Lambda=(10, 0, \cdots, 0)$ and $\Lambda=(0, 0, \cdots, 0)$ which correspond to exponential quadratic spline and usual quadratic spline, respectively.

Table. Observed errors at mesh points

x	exponential spline	usual spline
0.1	−0.253-4*)	0.167
0.2	0.119-4	−0.286-1
0.3	−0.205-5	0.491-2
0.4	0.351-6	−0.843-3
0.5	−0.602-7	0.145-3
0.6	0.103-7	−0.248-4
0.7	−0.177-8	0.426-5
0.8	0.304-9	−0.730-6
0.9	−0.507-10	0.122-6

*) We denote 0.253×10^{-4} by 0.253-4.

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