72. *Fourier Coefficients of Eisenstein Series of Degree 3*

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Our aim is to give explicit formulas of Eisenstein series for $Sp_3(\boldsymbol{Z})$ except the 2-Euler factor. In case of $Sp_2(\boldsymbol{Z})$ they are given in [5] and essentially in [1]. It is known, [6], that they are products of local densities of quadratic forms up to elementary factors. Thus we have only to evaluate local densities.

In this note, we assume that p is an odd prime, and $m(\geqq 4)$ is even natural number. Set

$$S=\begin{pmatrix} & 1_{m/2} \\ 1_{m/2} & \end{pmatrix} \qquad (1_{m/2}=\text{the identity matrix of degee } m/2).$$

We use the notation $\alpha_p(T,S)$ in [2] for local densities, and for simplicity we write $\alpha(T)$ for $\alpha_p(T,S)$. Denote by χ the quadratic residue symbol mod p.

Theorem. *Set* $d=(1-p^{-m/2})(1-p^{2-m})$, *and for a diagonal matrix* T *whose diagonal entries are* $\varepsilon_i p^{a_i}$ $(1\leq i\leq 3)$ *with* $\varepsilon_i\in \boldsymbol{Z}_p^\times$, $-1\leq a_1\leq a_2\leq a_3$, *set*

$$\gamma(T)=\alpha(p^2T)-(p^{3-m/2}+p^{5-m})\alpha(pT)+p^{8-3m/2}\alpha(T),$$

and $\chi(T)=1$, $\chi(-\varepsilon_1\varepsilon_2)$, $\chi(-\varepsilon_2\varepsilon_3)$ *or* $\chi(-\varepsilon_1\varepsilon_3)$ *according to* $a_1\equiv a_2\equiv a_3 \bmod 2$, $a_1\equiv a_2\not\equiv a_3 \bmod 2$, $a_1\not\equiv a_2\equiv a_3 \bmod 2$ *or* $a_1\not\equiv a_2\not\equiv a_3 \bmod 2$. *Then we have*

$$\gamma(T)/d=1+\chi(T)p^{(2-m/2)(a_1+a_2+a_3+6)}, \qquad \text{and}$$

1) *in case* $a_1\equiv a_2 \bmod 2$,

$$\begin{aligned}\alpha(T)/d=&\sum_{0\leq k\leq a_1}\Big(\sum_{0\leq i\leq (a_1+a_2)/2-k-1}p^{(5-m)i}\Big)p^{(3-m/2)k}\\&+p^{a_1/2+(5-m)a_2/2}\Big(\sum_{0\leq k\leq a_1}p^{(2-m/2)k}\Big)\Big(\sum_{0\leq j\leq [(a_3-a_2-1)/2]}p^{(4-m)j}\Big)\\&+\chi(-\varepsilon_1\varepsilon_2)p^{a_1/2+(5-m)a_2/2}\Big(\sum_{1\leq k\leq a_1+1}p^{(2-m/2)k}\Big)\Big(\sum_{0\leq j\leq [(a_3-a_2)/2-1]}p^{(4-m)j}\Big)\\&+\chi(T)p^{(a_1+a_2)/2+(2-m/2)a_3}\Big(\sum_{0\leq k\leq a_1}p^{(2-m/2)k}\Big)\Big(\sum_{0\leq j\leq (a_2-a_1)/2}p^{(3-m)j}\Big)\\&+\chi(T)p^{(m/2-1)a_1+(2-m/2)(a_2+a_3)+3-m}\sum_{0\leq k\leq a_1-1}\Big(\sum_{0\leq j\leq k}p^{(1-m/2)j}\Big)p^{(2-m/2)k},\end{aligned}$$

2) *in case* $a_1\not\equiv a_2 \bmod 2$,

$$\begin{aligned}\alpha(T)/d=&\sum_{0\leq k\leq a_1}\Big(\sum_{0\leq j\leq (a_1+a_2-1)/2-k}p^{(5-m)j}\Big)p^{(3-m/2)k}\\&+\chi(T)p^{(m/2-1)a_1+(2-m/2)(a_2+a_3)+3-m}\sum_{0\leq k\leq a_1-1}\Big(\sum_{0\leq j\leq k}p^{(1-m/2)j}\Big)p^{(2-m/2)k}\\&+\chi(T)p^{(a_1+a_2)/2+(2-m/2)a_3+(3-m)/2}\Big(\sum_{0\leq k\leq a_1}p^{(2-m/2)k}\Big)\Big(\sum_{0\leq j\leq (a_2-a_1-1)/2}p^{(3-m)j}\Big).\end{aligned}$$

Corollary 1. *Let* $a_k(T)$ *be the Fourier coefficient of Eisenstein series of weight* k $(\equiv 0 \bmod 2)$ *for* $Sp_n(\boldsymbol{Z})$ $(n\leq 3)$. *Let* T *be a half integral positive definite* $n\times n$ *matrix. Then the Dirichlet series*

$$\sum_{t=0}^{\infty} a_k(p^tT)p^{-ts}$$

is a rational function in p^{-s}, *and the denominator is*

$$(1-p^{-s})\prod_{r=1}^{n}(1-p^{rk-r(r+1)/2-s}),$$

and the degree of the numerator in p^{-s} *is* n, *and if* $(p, |T|)=1$, *then it is* $n-1$.

Corollary 2. *Let* T *be a half-integral positive definite* 3×3 *matrix. Then Siegel series* $b_p(s, T)$ *in* [3] *is equal to* $\alpha(T)$ *in the theorem when* m *is replaced by* $2s$.

To prove the theorem, we have only to see that the values in it satisfy the induction formula in Theorem 1 in [2] which is explicitly given in

Lemma. *Let* η *be any unit of* $Z_p^\times$ *with* $\chi(\eta)=-1$, *and let* $\tilde{S}$ *be a symmetric* Z_p-*integral matrix of degree* n *with* $|\tilde{S}|\neq 0$. *For a diagonal matrix* T *whose entries are* $\varepsilon_i p^{a_i}$ ($\varepsilon_i\in Z_p^\times$, $a_i\in Z$, $1\leq i\leq 3$) *we write* $\alpha(\varepsilon_1p^{a_1}, \varepsilon_2p^{a_2}, \varepsilon_3p^{a_3})$ *for* $\alpha_p(T, \tilde{S})$.

Set $X(\varepsilon_1p^{a_1}, \varepsilon_2p^{a_2}, \varepsilon_3p^{a_3})=\alpha(\varepsilon_1p^{a_1}, \varepsilon_2p^{a_2}, \varepsilon_3p^{a_3})-p^{4-n}\alpha(\varepsilon_1p^{a_1}, \varepsilon_2p^{a_2}, \varepsilon_3p^{a_3-2})$.

1) *In case* $0\leq a_1<a_2<a_3$.

$$X(\varepsilon_1p^{a_1}, \varepsilon_2p^{a_2}, \varepsilon_3p^{a_3})-p^{5-n}X(\varepsilon_1p^{a_1}, \varepsilon_2p^{a_2-2}, \varepsilon_3p^{a_3})$$
$$=d_p(T,\tilde{S})+p^{6-n}\{X(\varepsilon_1p^{a_1-2}, \varepsilon_2p^{a_2}, \varepsilon_3p^{a_3})-p^{5-n}X(\varepsilon_1p^{a_1-2}, \varepsilon_2p^{a_2-2}, \varepsilon_3p^{a_3})\}.$$

2) *In case* $0\leq a_1=a_2<a_3$ ($\varepsilon_1=1$ *can be supposed*).

$$X(p^{a_1}, \varepsilon_2p^{a_1}, \varepsilon_3p^{a_3})+p^{11-2n}X(p^{a_1-2}, \varepsilon_2p^{a_1-2}, \varepsilon_3p^{a_3})$$
$$=d_p(T,\tilde{S})+(1/2)p^{5-n}(p-1-\chi(\varepsilon_2)-\chi(-\varepsilon_2))X(p^{a_1-2}, \varepsilon_2p^{a_1}, \varepsilon_3p^{a_3})$$
$$+p^{5-n}X(\varepsilon_2p^{a_1-2}, p^{a_1}, \varepsilon_3p^{a_3})+(1+\chi(-\varepsilon_2))p^{5-n}X(p^{a_1-1}, \varepsilon_2p^{a_1-1}, \varepsilon_3p^{a_3})$$
$$+(1/2)p^{5-n}(p-1+\chi(\varepsilon_2)-\chi(-\varepsilon_2))X(\eta p^{a_1-2}, \eta\varepsilon_2p^{a_1}, \varepsilon_3p^{a_3}).$$

3) *In case* $0\leq a_1<a_2=a_3$ ($\varepsilon_2=1$ *can be supposed*). *Set*

$$Z(T)=\alpha(T)-p^{4-n}\alpha(\varepsilon_1p^{a_1}, \varepsilon_3p^{a_2-2}, p^{a_2})$$
$$-p^{4-n}(1+\chi(-\varepsilon_3))\alpha(\varepsilon_1p^{a_1}, p^{a_2-1}, \varepsilon_3p^{a_2-1})$$
$$-(1/2)p^{4-n}(p-1-\chi(-\varepsilon_3)-\chi(\varepsilon_3))\alpha(\varepsilon_1p^{a_1}, p^{a_2-2}, \varepsilon_3p^{a_2})$$
$$-(1/2)p^{4-n}(p-1-\chi(-\varepsilon_3)+\chi(\varepsilon_3))\alpha(\varepsilon_1p^{a_1}, \eta p^{a_2-2}, \eta\varepsilon_3p^{a_2})$$
$$+p^{9-2n}\alpha(\varepsilon_1p^{a_1}, p^{a_2-2}, \varepsilon_3p^{a_2-2}),$$

then

$$Z(T)-p^{6-n}Z(\varepsilon_1p^{a_1-2}, p^{a_2}, \varepsilon_3p^{a_2})=d_p(T,\tilde{S}).$$

4) *In case* $0\leq a_1=a_2=a_3=a$ ($\varepsilon_1=\varepsilon_2=1$ *can be supposed and set* $\varepsilon_3=\varepsilon$).

$$\alpha(T)-p^{15-3n}\alpha(p^{-2}T)$$
$$=d_p(T,\tilde{S})+(1/2)p^{4-n}(p^2+\chi(-\varepsilon)p-1-\chi(\varepsilon))\alpha(p^{a-2}, p^a, \varepsilon p^a)$$
$$+(1/2)p^{4-n}(p^2-\chi(-\varepsilon)p-1+\chi(\varepsilon))\alpha(\eta p^{a-2}, p^a, \eta\varepsilon p^a)$$
$$+p^{4-n}\alpha(\varepsilon p^{a-2}, p^a, p^a)+p^{4-n}(1+\chi(-1))\alpha(p^{a-1}, p^{a-1}, \varepsilon p^a)$$
$$+2p^{4-n}(1+\chi(-\varepsilon))\alpha(p^{a-1}, \varepsilon p^{a-1}, p^a)$$
$$+p^{4-n}(p-2-\chi(-1)-2\chi(-\varepsilon))\alpha(p^{a-1}, -p^{a-1}, -\varepsilon p^a)$$
$$-(1/2)p^{9-2n}(p^2-p+\chi(-1)p+\chi(-1))\alpha(p^{a-2}, p^{a-2}, \varepsilon p^a)$$
$$-(1/2)p^{9-2n}(p-\chi(-1))\alpha(p^{a-2}, \varepsilon p^{a-2}, p^a)$$
$$-(1/2)p^{9-2n}(p^2-p-\chi(-1)p+\chi(-1))\alpha(p^{a-2}, \eta p^{a-2}, \eta\varepsilon p^a)$$

$$-(1/2)p^{9-2n}(p-\chi(-1))\alpha(p^{a-2}, \varepsilon\eta p^{a-2}, \eta p^{a})$$
$$-p^{9-2n}(1+\chi(-1))\alpha(\varepsilon p^{a-2}, p^{a-1}, p^{a-1})$$
$$-(1/2)p^{9-2n}(1+\chi(-\varepsilon))(p-\chi(-1))\alpha(p^{a-2}, p^{a-1}, \varepsilon p^{a-1})$$
$$-(1/2)p^{9-2n}(1-\chi(-\varepsilon))(p-\chi(-1))\alpha(\eta p^{a-2}, p^{a-1}, \eta\varepsilon p^{a-1}).$$

When $\tilde{S}=S$, *we have*

$$d_p(T,S)=(1-p^{-m/2})(1-p^{2-m})\times\begin{cases}(1+p^{3-m/2})(1-p^{4-m}) & \text{if } 0<a_i\ (i=1,2,3),\\ 1-p^{4-m} & \text{if } 0=a_1<a_i\ (i=2,3),\\ 1+\chi(-\varepsilon_1\varepsilon_2)p^{2-m/2} & \text{if } 0=a_1=a_2<a_3,\\ 1 & \text{if } 0=a_1=a_2=a_3.\end{cases}$$

Remark. Let $\tilde{S}$ be a ternary unimodular matrix. Then $\alpha(p^4T)-(p^{6-n}+p^{10-2n})\alpha(p^2T)+p^{16-3n}\alpha(T)=0$ holds for every ternary symmetric matrix $T\in M_3(\boldsymbol{Z}_p)$ with $|T|\neq 0$ by [2] ($n=3$).

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