67. *Explosion Problems for Symmetric Diffusion Processes*

By Kanji ICHIHARA
Department of Mathematics, College of General Education, Nagoya University

(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 12, 1984)

§1. **Introduction.** Let L be a strictly elliptic partial differential operator with measurable coefficients of the form:

$$L=\frac{1}{b}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)$$

where (a_{ij}) is symmetric and $b>0$. We assume that for each non-empty compact subset K of $\boldsymbol{R}^n$, there exists a constant $\lambda=\lambda(K)$ such that

$$\lambda^{-1}|\xi|^2\leqq\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\leqq\lambda|\xi|^2$$

and

$$b(x)\leqq\lambda$$

for all x in K and ξ in $\boldsymbol{R}^n$. Then we can construct a unique minimal diffusion process $(X_t, \zeta, P_x)_{x\in\boldsymbol{R}^n}$ by using the theory of Dirichlet spaces, Fukushima [2] (see also Morrey [5]) where ζ is the explosion time of the process, i.e. $\lim_{t\nearrow\zeta(\omega)}|X_t(\omega)|=+\infty$ if $\zeta(\omega)<+\infty$. One of the basic problems for the diffusion processes is to find conditions for conservativeness and explosion. Such conditions for one dimensional diffusion processes have been established by Feller [1] in connection with the classification of boundary points. His conditions are given in terms of the scale and speed measures. In multidimensional cases, Hasminskii [3] has obtained sufficient conditions for conservativeness and explosion for diffusion processes which can be constructed by means of Itô's stochastic differential equations. Hasminskii's idea (see McKean [4]) can not be applied to our cases since the coefficients a_{ij} of the above operator L are not necessarily smooth. However we can use the theory of Dirichlet spaces to get conditions for conservativeness and explosion.

§2. **α-equilibrium potential and α-capacity** (Fukushima [2]). Let $\boldsymbol{B}_n$ be the closed unit ball $\{|x|\leqq 1\}$ in $\boldsymbol{R}^n$ and τ_0 the first hitting time of $\boldsymbol{B}_n$ by the process X_t. The α-equilibrium potential $e_\alpha(x)$ of $\boldsymbol{B}_n$ $(\alpha\geqq 0)$ is defined by

$$e_\alpha(x)=\begin{cases}E_x[e^{-\alpha\tau_0}] & \text{for } \alpha>0\\ P_x[\tau_0<\zeta] & \text{for } \alpha=0.\end{cases}$$

For u, v in the space $C_0^\infty(\boldsymbol{B}^n)$ of infinitely differentiable, real valued functions with compact support, set

$$\mathcal{E}_\alpha(u,v)=\sum_{i,j=1}^{n}\int_{\boldsymbol{R}^n} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx+\alpha\int_{\boldsymbol{R}^n} buvdx.$$

Denote the $\mathcal{E}_\alpha$-closure of $C_0^\infty(\boldsymbol{R}^n)$ by $\mathcal{F}_\alpha$. $(\mathcal{E}_\alpha, \mathcal{F}_\alpha)$ for $\alpha>0$ is a Hilbert space. It is also known that $(\mathcal{E}_0, \mathcal{F}_0)$ is a Hilbert space if and only if X_t is transient.

The α-capacity $C_\alpha(\boldsymbol{B}_n)$ of $\boldsymbol{B}_n$ is defined as follows. For $\alpha>0$,

$$C_\alpha(\boldsymbol{B}_n)=\inf_{\substack{u\in\mathcal{F}_\alpha\\ u\geqq 1 \text{ a.e. on } \boldsymbol{B}_n}} \mathcal{E}_\alpha(u,u).$$

For $\alpha=0$ and X_t being transient,

$$C_0(\boldsymbol{B}_n)=\inf_{\substack{u\in\mathcal{F}_0\\ u\geqq 1 \text{ a.e. on } \boldsymbol{B}_n}} \mathcal{E}_0(u,u).$$

Note that for each $\alpha>0$, $e_\alpha\in\mathcal{F}_\alpha$ and

$$C_\alpha(\boldsymbol{B}_n)=\mathcal{E}_\alpha(e_\alpha, e_\alpha)>0.$$

Moreover, when X_t is transient, $e_0\in\mathcal{F}_0$ and

$$C_0(\boldsymbol{B}_n)=\mathcal{E}_0(e_0, e_0)>0.$$

§ 3. **Main results.** **Theorem 1.** *The following statements are equivalent*:

(1) X_t *is conservative*, i.e. $P_\cdot[\zeta=+\infty]=1$ *on* $\boldsymbol{R}^n$.

(2) *For some* $\alpha>0$

$$\int_{\boldsymbol{R}^n}\alpha e_\alpha(x)b(x)dx=C_\alpha(\boldsymbol{B}_n).$$

(3) *For all* $\alpha>0$

$$\int_{\boldsymbol{R}^n}\alpha e_\alpha(x)b(x)dx=C_\alpha(\boldsymbol{B}_n).$$

It should be remarked that in general we have

$$\int_{\boldsymbol{R}^n}\alpha e_\alpha(x)b(x)dx\leqq C_\alpha(\boldsymbol{B}_n)$$

for every $\alpha>0$.

Theorem 2. *If* X_t *is transient, then the limit of*

$$\int_{\boldsymbol{R}^n}\alpha e_\alpha(x)b(x)dx$$

as $\alpha\downarrow 0$ *exists and*

$$\lim_{\alpha\downarrow 0}\int_{\boldsymbol{R}^n}\alpha e_\alpha(x)b(x)dx\begin{cases}=C_0(\boldsymbol{B}_n), & \textit{when } P_\cdot[\zeta=+\infty]=1 \quad \textit{on } \boldsymbol{R}^n\\ \in(0, C_0(\boldsymbol{B}_n)), & \textit{when } 0<P_\cdot[\zeta=+\infty]<1 \quad \textit{on } \boldsymbol{R}^n\\ =0, & \textit{when } P_\cdot[\zeta=+\infty]=0 \quad \textit{on } \boldsymbol{R}^n.\end{cases}$$

To find sufficient conditions on coefficients a_{ij}, b for conservativeness and explosion, define

$$A_+(r)=\int_{S^{n-1}}(A(r\sigma)\sigma,\sigma)d\sigma \qquad \text{for } r>0,$$

$$B_+(r)=\sup_{\sigma\in S^{n-1}}\left\{\frac{(A(r\sigma)\sigma,\sigma)}{b(r\sigma)}\right\} \qquad \text{for } r>0,$$

where $A(x)=(a_{ij}(x))_{i,j=1,\dots,n}$ and $d\sigma$ is the uniform measure on S^{n-1}.

Theorem 3. (1) *If for some* $\alpha>0$

$$\lim_{r\nearrow+\infty}\frac{\exp\left\{-\int_1^r\frac{2\sqrt{\alpha}}{\sqrt{B_+(s)}}ds\right\}}{\int_r^\infty s^{1-n}A_+(s)^{-1}ds}=0,$$

then the process X_t *is conservative.*

(2) *If*

$$\int_{S^{n-1}}d\sigma\int_1^\infty r^{1-n}(A(r\sigma)^{-1}\sigma,\sigma)\left[\int_1^r s^{n-1}b(s\sigma)ds\right]^2dr<+\infty$$

where $A(x)^{-1}$ *is the inverse matrix of* $A(x)$, *then the explosion is sure,* i.e. $P_{\cdot}[\zeta<+\infty]=1$ *on* $\boldsymbol{R}^n$.

The proof of Theorem 3 is based on estimates of some Dirichlet integrals of e_α. The details of the above results together with some extensions will appear elsewhere.

References

[1] W. Feller: The parabolic differential equations and the associated semi-groups of transformations. Ann. of Math., **55**, 468–519 (1952).

[2] M. Fukushima: Dirichlet Spaces and Markov Processes. Kodansha, Tokyo (1980).

[3] R. Z. Hasminskii: Ergodic properties of recurrent diffusion processes and stabilization of the problem to the Cauchy problem for parabolic equations. T. V., **5**, 196–214 (1960).

[4] H. P. McKean: Stochastic Integrals. Academic Press, N.Y. (1969).

[5] C. B. Morrey: Second order elliptic equations in several variables and Hölder continuity. Math. Z., **72**, 146–164 (1959).