1. A Short Proof of a Theorem Concerning Homeomorphisms of the Unit Circle*)

By Shinzô KAWAMURA
Department of Mathematics, Yamagata University

(Communicated by Kôsaku YOSIDA, M. J. A., Jan. 12, 1984)

1. In [4], Rieffel classified the C^*-algebras associated with irrational rotations on the unit circle S^1 in the complex plane. Recently these C^*-algebras have played an important rôle in the theory of operator algebras.

The author and Takemoto [1] extended the Rieffel's result to the case of C^*-algebras associated with monothetic compact abelian groups. A compact abelian group G is said to be monothetic if there exists a homomorphism from the group Z of all integers to a dense subgroup of G (cf. [5, 2.3]). In [1] and [2], we considered more general cases. Namely, we studied the C^*-algebras associated with topologically transitive compact dynamical systems. A dynamical system (Ω, σ) is said to be topologically transitive if the homeomorphism σ admits a point ω in the compact space Ω such that the orbit $O(\omega)$ $(=\{\sigma^n(\omega): n \in Z\})$ is dense in Ω (cf. [6, 5.4]). So we are interested in the existence and the classification of such dynamical systems. In case $\Omega = S^1$, every topologically transitive homeomorphism σ is conjugate to an irrational rotation. It is well-known that this theorem was established by Poincaré [2]. Nowadays we can see several kinds of proofs in many books, in which the rotation number of σ plays an important rôle. In this note, we give a short and elementary proof without rotation numbers.

2. Two homeomorphisms σ_1 and σ_2 of S^1 are said to be conjugate if there exists a homeomorphisms h on S^1 such that $\sigma_1 = h\sigma_2 h^{-1}$. For a real number θ, we denote by R_θ the rotation: $R_\theta(e^{2\pi i x}) = e^{2\pi i (x+\theta)}$ on S^1. We shall prove the following equivalences.

Theorem. *Let σ be a homeomorphism of S^1. Then the following statements are equivalent;*

(1) *$O(z)$ is dense in S^1 for some z in S^1,*

(2) *$O(z)$ is dense in S^1 for every z in S^1,*

(3) *σ is conjugate to R_θ for some irrational number θ* $(0<\theta<1/2)$.

When σ satisfies the condition (1) *or* (2), *the rotation R_θ in* (3) *is uniquely determined.*

*) Partly supported by the Grant-in-Aid for Scientific Research (No. 58540056), the Ministry of Education, Science and Culture, Japan.

For the proof of equivalences, we have only to show the implication (1)$\Rightarrow$(3). To prove this, we need the following lemma.

Lemma. *Let σ be a homeomorphism of S^1 satisfying the condition (1) in the theorem. Then, for any integer k, σ^k admits no fixed point.*

Proof. We first suppose that there exists a fixed point ω for σ. Then the restriction σ_0 of σ to $S^1-\{\omega\}$ can be regarded as a homeomorphism of the real line $\boldsymbol{R}$. Hence, for any z in S^1, $O(z)$ is not dense in S^1.

Next, we suppose that there exists a fixed point ω for σ^k $(k\neq\pm 1)$. For z_1 and z_2 in S^1, we denote by (z_1, z_2) the set $\{e^{2\pi i x}\,|\,x_1<x<x_2\}$, where x_1 and x_2 are the real numbers determined by the relation: $z_1=e^{2\pi i x_1}$ $(0\leqq x_1<1)$ and $z_2=e^{2\pi i x_2}$ $(x_1\leqq x_2<x_1+1)$. Then the open interval $(\omega, \sigma(\omega))$ is mapped onto itself or $(\sigma(\omega), \omega)$ by σ^k and the restriction of σ^k to $(\omega, \sigma(\omega))$ becomes an order-isomorphism or an anti-order-isomorphism, where the order in the interval (z_1, z_2) is considered as the one induced by the usual order in (x_1, x_2). Therefore, σ^{2k} becomes an order-isomorphism of $(\omega, \sigma(\omega))$ onto itself, and we can assume that the point z satisfying the condition (1) belongs to this interval. Hence the set $\{\sigma^{2nk}(z)\,|\,n\in Z\}$ has exactly two limit points in S^1. Since

$$O(z)=\bigcup_{i=0}^{2k-1}\{\sigma^{2nk+i}(z)\,|\,n\in Z\},$$

$O(z)$ has only finite limit points in S^1. This means that the closure of $O(z)$ does not coincide with S^1. Q.E.D.

Proof of Theorem. (1)$\Rightarrow$(3). Let y and y_0 be the real numbers in $[0, 1)$ determined by the relations: $z=e^{2\pi i y}$ and $\sigma(e^{2\pi i 0})=e^{2\pi i y_0}$. Since σ admits no fixed point in S^1, there exists a homeomorphism f of $[0, 1)$ onto $[y_0, y_0+1)$ such that $\sigma(e^{2\pi i x})=e^{2\pi i f(x)}$. We extend f to a homeomorphism F of the real line $\boldsymbol{R}$ onto itself as follows;

$$F(x+n)=f(x)+n \qquad (0\leqq x<1,\ n\in \boldsymbol{Z}).$$

Furthermore we consider the homeomorphism T of $\boldsymbol{R}$ defined by the translation: $x\rightarrow x+1$ $(x\in\boldsymbol{R})$. We denote by G the subgroup of all homeomorphisms of $\boldsymbol{R}$ generated by F and T. Put $G_1=\{F^pT^q(y)\,|\,p, q\in\boldsymbol{Z}\}$. The set G_1 is dense in $\boldsymbol{R}$ because $O(z)$ is dense in S^1, and the order in the subset G_1 of the real line induces an order in G. Namely we define an order $\preceq$ in G as follows; for p, p', q and q' in $\boldsymbol{Z}$,

$$F^pT^q\preceq F^{p'}T^{q'} \qquad \text{if} \quad F^pT^q(y)\leqq F^{p'}T^{q'}(y).$$

Then $(G, \preceq)$ becomes a totally ordered abelian group. Moreover we can show that the order $\preceq$ is archimedian. Suppose that $F^pT^q\succ I$, that is, $F^pT^q(y)>I(y)=y$. Since F^p and T^q are monotonic increasing functions on $\boldsymbol{R}$, so is $H=F^pT^q$. Since $\sigma^p(e^{2\pi i x})=e^{2\pi i H(x)}$ $(0\leqq x<1)$, the preceding lemma implies that there is a non-zero distance between the graph of H and the diagonal line in $\boldsymbol{R}\times\boldsymbol{R}$. Since $H(y)>y$, there exists a positive number ε such that $H(x)>x+\varepsilon$ for all x in $\boldsymbol{R}$. Thus $H^k(y)$

$>y+k\varepsilon$. Therefore, for any element $g \succ I$, there exists a natural number k such that $H^k \succ g$. We know that every archimedian totally ordered abelian group is order-isomorphic to a subgroup of $\boldsymbol{R}$ ([5, 8.1.2]). Hence $(G, \preceq)$ is order-isomorphic to a subgroup G_2 of $\boldsymbol{R}$ by an order isomorphism Φ, and we assume that $\Phi(T)=1$. Since the subset $\{F^pT^q(y) \mid F^pT^q(y)>y\}$ of G_1 does not have the smallest element, the subset $\{F^pT^q \mid F^pT^q \succ I\}$ of G does not, either. Hence the set of positive real numbers in G_2 does not have the smallest element, so that G_2 is a dense subgroup of $\boldsymbol{R}$. Since $\Phi(F^pT^q)=p\Phi(F)+q$, we have $G_2=\{p\Phi(F)+q \mid p, q \in \boldsymbol{Z}\}$. Hence $\Phi(F)$ must be an irrational number θ and belongs to the open interval $(0, 1)$, because $0=\Phi(I)<\Phi(F)<\Phi(T)=1$. Since both G_1 and G_2 are dense in $\boldsymbol{R}$, the order-isomorphism $k' : F^pT^q(y) \to p^\theta+q$ $(G_1 \to G_2)$ can be extended to the unique homeomorphism k of $\boldsymbol{R}$. We put $h(e^{2\pi ix})=e^{2\pi ik(x)}$ if $0<\theta<1/2$. Then it follows that $h\sigma h^{-1}=R_\theta$. In the case where $1/2<\theta<1$, putting $h(e^{2\pi ix})=e^{-2\pi ik(x)}$, we have $h\sigma h^{-1}=R_{1-\theta}$. The uniqueness of R_θ is easily shown by seeing the set $\{e^{2\pi in\theta} \mid n \in \boldsymbol{Z}\}$ of eigenvalues of R_θ (cf. [6, Definition 5.8]). Q.E.D.

Acknowledgement. The author is grateful to his colleague S.-I. Watanabe for helpful discussions.

References

[1] S. Kawamura and H. Takemoto: C^*-algebras associated with shift dynamical systems (to appear in J. Math. Soc. Japan, **36** (1984)).
[2] ——: On the classification of C^*-algebras associated with shift dynamical systems (to appear in the Proceedings of the Japan-U.S. Seminar on Geometric Methods in Operator Algebras (1983)).
[3] H. Poincaré: Mémoire sur les courbes définies par une équation différentielle. Oeuvres t.1 (1928).
[4] M. Rieffel: C^*-algebras associated with irrational rotations. Pacific J. Math., **93**, 415–429 (1981).
[5] W. Rudin: Fourier Analysis on Groups. Interscience Publ. (1962).
[6] P. Walter: An introduction to ergodic theory. G.T.M. **79**, Springer Verlag, New York-Heidelberg-Berlin (1982).