26. An Application of the Perturbation Theorem for m-Accretive Operators

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The present note is concerned with the semi-linear equation $-\Delta u(x)+\beta(x,u(x))=f(x)$ on the whole space $\boldsymbol{R}^n$. This equation was recently treated by Sohr [4]. As an application of the perturbation theorem in Okazawa [3] we shall improve the result obtained in [4]. Here, it should be noted that a quite general theorem has been established by Brezis-Crandall-Pazy [1] and Konishi [2] if β does not explicitly depend on x: $\beta(x,u)=\beta(u)$.

1. Preliminaries. We consider only real-valued functions. An operator A (with domain $D(A)$ and range $R(A)$) in $L^2=L^2(\boldsymbol{R}^n)$ is said to be *accretive* (or monotone) if

$$(Au-Av,\ u-v)\geqq 0 \qquad \text{for } u,\ v\in D(A).$$

We say that an accretive operator A is *m-accretive* if $R(1+\lambda A)=L^2$ for some (and hence for every) $\lambda>0$. The *Yosida approximation* $\{B_\varepsilon\}$ of an m-accretive operator B is defined by

$$B_\varepsilon=\varepsilon^{-1}[1-(1+\varepsilon B)^{-1}], \qquad \varepsilon>0.$$

The following lemma is a Hilbert space case of Lemma 6.2 in [3].

Lemma 1. *Let A and B be m-accretive operators in L^2, with $D(A)\cap D(B)$ non-empty. Assume that there exist a constant b $(0\leqq b<1)$ and a nondecreasing function $\psi(r)\geqq 0$ of $r\geqq 0$ such that for all $u\in D(A)$ and $\varepsilon>0$,*

$$(Au, B_\varepsilon u)\geqq -\psi(\|u\|)-b\|B_\varepsilon u\|^2.$$

Then $A+B$ is also m-accretive in L^2.

Now, let J be an open interval on $\boldsymbol{R}$ and β be a real-valued function of class $C^1(\boldsymbol{R}^n\times J)$:

$$\beta(x,s)=\beta(x_1,x_2,\cdots,x_n,s).$$

We assume that $0\in J$ and

(i) $\beta(x,0)=0$ for every $x\in\boldsymbol{R}^n$, and $\partial\beta/\partial s\geqq 0$ on $\boldsymbol{R}^n\times J$.

Then we can introduce the following accretive operator $\tilde\beta$ in L^2:

$$D(\tilde\beta)=\{u\in L^2;\ u(x)\in J(\text{a.e.}),\ \beta(x,u(x))\in L^2\},$$
$$\tilde\beta u(x)=\beta(x,u(x)) \qquad \text{for } u\in D(\tilde\beta).$$

Lemma 2. *Let $\tilde\beta$ be the accretive operator as above. Then $\tilde\beta$ is m-accretive if*

(ii) *for every $x\in\boldsymbol{R}^n$, $\beta(x,\cdot): J\to\boldsymbol{R}$ is onto.*

Proof. Since $\tilde\beta$ is closed, it suffices to show that $R(\tilde\beta+1)$ contains a dense subset of L^2. To see this, let $v \in C_0^1(\boldsymbol{R}^n)$. Then by the implicit function theorem the equation

$$\beta(x, s)+s=v(x)$$

has a unique solution $u \in C_0^1(\boldsymbol{R}^n)$ such that

$$\beta(x, u(x))+u(x)=v(x).$$

Consequently, $R(\tilde{\beta}+1)$ contains $C_0^1(\boldsymbol{R}^n)$. Q.E.D.

2. Semi-linear equations. Let A be the minus Laplacian in $L^2 = L^2(\boldsymbol{R}^n)$: $Au(x)=-\Delta u(x)$ for $u \in H^2(\boldsymbol{R}^n)$. Then A is a nonnegative selfadjoint operator in L^2. In other words, A is symmetric and m-accretive.

Setting $B=\tilde{\beta}$, the main theorem in this note is stated as follows.

Theorem 3. *Let A and B be m-accretive operators as above; namely, assume that conditions* (i) *and* (ii) *are satisfied. Assume further that there are nonnegative constants a and b ($b<4$) such that for all $(x, s) \in \boldsymbol{R}^n \times J$,*

$$\sum_{j=1}^{n}\left|\frac{\partial\beta}{\partial x_j}(x, s)\right|^2 \leqq \{as^2+b[\beta(x, s)]^2\}\frac{\partial\beta}{\partial s}(x, s). \tag{1}$$

Then $A+B=-\Delta+\tilde{\beta}$ with domain $H^2(\boldsymbol{R}^n)\cap D(\tilde{\beta})$ is m-accretive in L^2.

Proof. It suffices to show that $A+B+1$ is m-accretive. So, we may assume that $\partial\beta/\partial s \geqq 1$ on $\boldsymbol{R}^n \times J$. In fact, $\beta(x, s)$ in (1) can be replaced by $\beta(x, s)+s$.

The proof is based on Lemma 1. Indeed, we can show that

$$4(Au, B_\varepsilon u) \geqq -a\|u\|^2-b\|B_\varepsilon u\|^2 \qquad \text{for all } u \in D(A). \tag{2}$$

Let $u \in C_0^\infty(\boldsymbol{R}^n)$. Setting $w(x)=(1+\varepsilon B)^{-1}u(x)$, we see from the implicit function theorem that $w \in C_0^1(\boldsymbol{R}^n)$ and

$$\frac{\partial w}{\partial x_j}(x)=-\left[1+\varepsilon\frac{\partial\beta}{\partial s}(x, w(x))\right]^{-1}\left[\varepsilon\frac{\partial\beta}{\partial x_j}(x, w(x))-\frac{\partial u}{\partial x_j}(x)\right].$$

So, we have

$$\begin{aligned}(Au, B_\varepsilon u)&=-\int_{\boldsymbol{R}^n}\Delta u(x)\cdot\varepsilon^{-1}[u(x)-w(x)]dx\\&=\varepsilon^{-1}\int_{\boldsymbol{R}^n}\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\left(\frac{\partial u}{\partial x_j}-\frac{\partial w}{\partial x_j}\right)dx\\&=\int_{\boldsymbol{R}^n}\left(1+\varepsilon\frac{\partial\beta}{\partial s}\right)^{-1}\frac{\partial\beta}{\partial s}\sum_{j=1}^{n}\left|\frac{\partial u}{\partial x_j}\right|^2dx\\&\quad+\int_{\boldsymbol{R}^n}\left(1+\varepsilon\frac{\partial\beta}{\partial s}\right)^{-1}\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\frac{\partial\beta}{\partial x_j}dx.\end{aligned}$$

Therefore, we obtain

$$(Au, B_\varepsilon u) \geqq -\frac{1}{4}\int_{\boldsymbol{R}^n}\left(\frac{\partial\beta}{\partial s}\right)^{-1}\sum_{j=1}^{n}\left|\frac{\partial\beta}{\partial x_j}\right|^2dx.$$

Since $C_0^\infty(\boldsymbol{R}^n)$ is a core of A, (2) follows from (1); note that $\beta(x, w(x)) = B_\varepsilon u(x)$ and $\|w\|=\|(1+\varepsilon B)^{-1}u\| \leqq \|u\|$. Q.E.D.

Corollary 4. *In Theorem 3 assume further that there is a con-*

stant $K>0$ *such that* $\partial\beta/\partial s \geqq K$ *on* $\boldsymbol{R}^n \times J$. *Then for every* $f \in L^2$ *there exists a unique solution* $u \in H^2(\boldsymbol{R}^n) \cap D(\tilde{\beta})$ *of the equation*

$$-\Delta u(x) + \beta(x, u(x)) = f(x).$$

This corollary improves Theorem 3.1 in Sohr [4] in which it is assumed that $\beta \in C^2(\boldsymbol{R}^n \times J)$ and $a=0$ in (1).

References

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