24. A Compact-Like Space which does not have a Countable Cover by C-Scattered Closed Subsets

By Tsugunori NOGURA
Department of Mathematics, Ehime University

(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1983)

Let K denote a class of spaces which are hereditary with respect to closed subspaces. Let FK denote the class of all $X=\bigcup\{X_m : m\leq n\}$, where X_m is a closed subset of X for each $m\leq n$, $n\in N$ (N denotes the natural numbers) and $X_m\in K$. Let C denote the class of all compact spaces. Then $FC=C$.

The topological game $G(K,X)$ is introduced and studied by R. Telgársky ([1], [2]). We use the notations in [1]. Each space considered here is assumed to be completely regular.

The following theorems are proved by R. Telgársky:

(a) ([1], Theorem 11.1). Let X be a hereditarily paracompact K-like ([1], p. 195) space. Then $X=\bigcup\{X_n : n\in N\}$, where X_n is a closed FK-scattered subset of X for each $n\in N$.

(b) ([2], Theorem 1.3). Let X be a K-like space. Then $X=\bigcup\{X_n : n\in N\}$, where X_n is a K-scattered subset of X for each $n\in N$.

(c) ([2], Remark 1.5). Let X be a K-like space. Assume that each open subset of X is the union of a σ-locally finite family of closed sets (in particular, that X is totally normal or hereditarily paracompact), then $X=\bigcup\{X_n : n\in N\}$, where X_n is a closed FK-scattered subset of X for each $n\in N$.

The following problem is posed by R. Telgársky ([2], Remark 1.5):

Does each K-like space have a countable cover by K-scattered closed subset?

The following simple example gives a negative answer to the above problem.

Theorem. (CH) *There exists a compact-like space X which does not have a countable cover by C-scattered closed subsets.*

Proof. Let $I=[0,1]$ be the closed unit interval. Well-order $I=\{x_\alpha : \alpha<\omega_1\}$, where ω_1 is the first uncountable ordinal number. Let $[0,\omega_1)$ ($[0,\omega_1]$) be the space of ordinal numbers less than (less than or equal to) ω_1 with the interval topology. For each $\alpha<\omega_1$, put $M_\alpha=\{(\alpha,x_\beta)\in[0,\omega_1)\times I,\ \beta\leq\alpha\}$ and $X=\bigcup\{M_\alpha : \alpha<\omega_1\}\cup\{\omega_1\}\times I$. We will show that the subspace X of the space $[0,\omega_1]\times I$ has desired properties. First we show that X is compact-like. Put $E_0=X$, $E_1=\{\omega_1\}\times I$. Let

E_2 be a closed subset of X such that $E_2 \cap E_1 = \phi$. Then there exists $\alpha < \omega_1$ such that $E_2 \subset ([0, \alpha] \times I) \cap X$. Since $([0, \alpha] \times I)$ X is countable, E_2 is compact-like ([1], Theorems 4–7). Let t be a winning strategy in $G(C, E_2)$. Put $s(E_0) = E_1$ and $s(E_0, E_1, \cdots, E_{2n}) = t(E_2, E_3, \cdots, E_{2n})$. Then s is a winning strategy in the game $G(C, X)$.

Next we show that X does not have a countable cover by C-scattered closed subsets. Assume $X = \bigcup \{S_n : n \in N\}$, where S_n is a C-scattered closed subset of X for each $n \in N$. Note that: For each $x \in I$, there exists $n \in N$ such that $(\omega_1, x) \in \mathrm{Cl}_X\{S_n \cap ([0, \omega_1) \times \{x\})\}$. Assume $(\omega_1, x) \notin \mathrm{Cl}_X\{S_n \cap ([0, \omega_1) \times \{x\})\}$, then there exists $\beta_n < \omega_1$ such that $S_n \cap ([\beta_n, \omega_1] \times \{x\}) = \phi$ for each $n \in N$. Put $x = x_\alpha$ and $\beta = \sup \{\{\beta_n : n \in N\} \cup \{\alpha\}\}$. Since $\beta < \omega_1$, $[\beta, \omega_1) \times \{x\}$ is a non-empty subset of X. This is a contradiction since $\{S_n : n \in N\}$ is a cover of X. The proof of the above note is completed. Put $I_n = \{x \in I : (\omega_1, x) \in \mathrm{Cl}_X\{S_n \cap ([0, \omega_1) \times \{x\})\}\}$. Then $I = \bigcup \{I_n : n \in N\}$ by the above note. We show that I_n is nowhere dense in I for each $n \in N$. Assume there exists $n \in N$ such that $\mathrm{Int}_I \mathrm{Cl}_I I_n \neq \phi$. Let $\{P_m : m \in N\}$ be a subset of I_n such that $\{p_m : m \in N\}$ is dense in $\mathrm{Int}_I \mathrm{Cl}_I I_n$. Put $A_m = S_n \cap ([0, \omega_1) \times \{p_m\})$ for each $m \in N$. Then $(\omega_1, p_m) \in \mathrm{Cl}_X A_m$ for each $m \in N$. Choose $(\alpha_1^m, p_m) \in A_m$ and $\alpha_1 < \omega_1$ such that $\alpha_1^m < \alpha_1$ for each $m \in N$. Choose $(\alpha_2^m, p_m) \in A_m$ and $\alpha_2 < \omega_1$ such that $\alpha_1 < \alpha_2^m < \alpha_2$ for each $m \in N$. Continuing in this manner we can get a sequence $\{\alpha_k : k \in N\}$ such that $\alpha_k < \alpha_k^m < \alpha_{k+1} < \omega_1$ for each $m \in N$ and $(\alpha_k^m, p_m) \in A_m$ for each $m \in N$. Put $\gamma = \sup \{\alpha_k : k \in N\}$. Since S_n is closed and $(\alpha_k^m, p_m) \in S_n$, $(\gamma, p_m) \in S_n$ for each $m \in N$. Since $X \cap (\{\gamma\} \times I)$ is countable, $\mathrm{Cl}_X\{(\gamma, p_m) : m \in N\}$ is a countable and closed subset of S_n which is homeomorphic to the space of rationals. This implies that S_n is not C-scattered. This is a contradiction. Thus I_n is nowhere dense in I. But this contradicts a well-known theorem of Baire. The proof is completed.

References

[1] R. Telgársky: Spaces defined by topological games. Fund. Math., **88**, 193–223 (1975).

[2] ——: ditto. II. ibid. (accepted in 1980).