21. The Nonrelativistic Limit of Modified Wave Operators for Dirac Operators

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We shall consider the Dirac operator

$$L(c)=c\sum_{j=1}^{3}\alpha_j D_j+c^2\beta+V(x)\qquad\left(x\in \boldsymbol{R}^3,\ D_j=\frac{1}{i}\frac{\partial}{\partial x_j}\right),$$

where $c>0$ is the velocity of light and α_j, β are 4×4 matrices given by

$$\alpha_1=\begin{bmatrix} & & & 1\\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix},\quad \alpha_2=\begin{bmatrix} & & & -i\\ & & i & \\ & -i & & \\ i & & & \end{bmatrix},\quad \alpha_3=\begin{bmatrix} & & 1 & \\ & & & -1\\ 1 & & & \\ & -1 & & \end{bmatrix},$$

$$\beta=\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1\end{bmatrix},$$

which satisfy the anti-commutation relation $\alpha_j\alpha_k+\alpha_k\alpha_j=2\delta_{jk}I$ ($j, k=1,2,3,4$) with $\alpha_4=\beta$ (I is the 4×4 unit matrix). The scalar potential $V(x)$ is assumed to satisfy the following *condition* (A); *there exist positive constants* $0<\delta$ (≤ 1), $e>0$ *and a positive integer* $m\geq 3$ *such that*

(A-1) $\quad m=3$ *if* $\delta>\frac{1}{2}$ *and* $m=\left[\frac{1}{\delta}\right]+3$ *if* $0<\delta\leq\frac{1}{2}$,

and $V(x)$ *is a real-valued* C^m*-function in* $\boldsymbol{R}^3\backslash 0$ *satisfying*

(A-2) $\quad D^\alpha V(x)=O(|x|^{-|\alpha|-\delta})$ *as* $|x|\to\infty$ $\quad(|\alpha|\leq m)$,

(A-3) $\quad |V(x)|\leq\frac{e}{r}$ $\quad(0<r\leq 1)$,

where $D^\alpha=D_1^{\alpha_1}D_2^{\alpha_2}D_3^{\alpha_3}$, $|\alpha|=\alpha_1+\alpha_2+\alpha_3$ *for a multi-index* $\alpha=(\alpha_1,\alpha_2,\alpha_3)\in \boldsymbol{Z}^3$ $(\alpha_j\geq 0)$.

It is evident that $L(c)$ is formally selfadjoint in the Hilbert space $\mathcal{L}^2=[L^2(\boldsymbol{R}^3)]^4$. A symmetric operator $L(c)$ defined on $[C_0^\infty(\boldsymbol{R}^3)]^4$ has the (not necessarily unique) selfadjoint extension[1], and is essentially selfadjoint if $c>2e$ (see Kato [7, Chapter V, § 4] and note that Arai [3] proposes a refined result). We denote by $H_0(c)$ the unperturbed selfadjoint operator with $V(x)\equiv 0$.

Let $H_0(c)=\int_{-\infty}^{\infty}\lambda dE^{(c)}(\lambda)$ be the spectral representation of $H_0(c)$. It

[1] This fact will be also proved elsewhere.

is known that $H_0(c)$ is absolutely continuous and $\sigma(H_0(c))=(-\infty, -c^2] \cup [c^2, \infty)$. Let us define

$$P_j(c)=\frac{1}{2}\int_{-\infty}^{\infty}\left(I+\tau_j\frac{\lambda}{|\lambda|}\right)dE^{(c)}(\lambda) \qquad (j=1,2),$$

$$\tau_1=1, \qquad \tau_2=-1.$$

Then $P_1(c)$ ($P_2(c)$) is the spectral projection to the positive (negative) spectrum, satisfying

$$P_1(c)+P_2(c)=I.$$

Moreover we have

$$(1) \qquad (P_j(c)f)^{\wedge}(\xi)=P_j(c,\xi)\hat{f}(\xi)=\frac{1}{2}\left(I+\tau_j\frac{\sum_{k=1}^{3}\alpha_k\xi_k+c\beta}{\sqrt{|\xi|^2+c^2}}\right)\hat{f}(\xi)$$

for $f \in \mathcal{L}^2$, and

$$(2) \qquad (H_0(c)P_j(c)f)^{\wedge}(\xi)=\tau_j c\sqrt{|\xi|^2+c^2}P_j(c,\xi)\hat{f}(\xi)$$

for $f \in D(H_0(c))$, where $\hat{f}(\xi)$ is the Fourier transform of $f(x)$ defined by

$$(\mathcal{F}f)(\xi)=\hat{f}(\xi)=(2\pi)^{-3/2}\int_{\mathbf{R}^3} e^{-i\langle x,\xi\rangle}f(x)dx.$$

Let $H(c)$ be a selfadjoint extension of $L(c)$ on $[C_0^\infty(\mathbf{R}^3)]^4$. If the potential $V(x)$ satisfies

$$V(x)=O(|x|^{-\delta}) \quad (|x|\to\infty), \quad \delta>1,$$

the wave operator

$$W_\pm^0(c)=\text{s-lim}_{t\to\pm\infty}\exp(itH(c))\exp(-itH_0(c))$$

exists under some additional conditions (Prosser [8]). If $0<\delta\le 1$, however, we shall not always expect the existence of $W_\pm^0(c)$ without modifying $\exp(-itH_0(c))$ appropriately, as is shown for Schrödinger operators (Dollard [6]).

Noting that

$$(\exp(-itH_0(c))f)^{\wedge}(\xi)=\exp(-itc\sqrt{|\xi|^2+c^2})P_1(c,\xi)\hat{f}(\xi) + \exp(itc\sqrt{|\xi|^2+c^2})P_2(c,\xi)\hat{f}(\xi)$$

for $f \in \mathcal{L}^2$ in view of (2), we define

$$\exp(-iX_j(t,c))f=\mathcal{F}^{-1}(\exp(-iX_j(t,c,\xi))\hat{f}(\xi)),$$

for $f \in \mathcal{L}^2$, where

$$X_j(t,c,\xi)=\tau_j ct\sqrt{|\xi|^2+c^2}+Z_j^{(n)}(t,c,\xi)$$

$$Z_j^{(k)}(t,c,\xi)=\int_{\operatorname{sgn} t}^{t} V\left(\tau_j\frac{cs\xi}{\sqrt{|\xi|^2+c^2}}+\operatorname{grad}_\xi Z_j^{(k-1)}(s,c,\xi)\right)ds,$$

$$Z_j^{(0)}(t,c,\xi)=0, \qquad n=[1/\delta],$$

$$\operatorname{sgn} t=1\ (t>0), \ =-1\ (t<0).$$

The idea of the choice $X_j(t,c,\xi)$ is suggested by Alsholm-Kato [2] and Buslaev-Matveev [4].

Theorem 1. *Assume that $V(x)$ satisfies the condition* (A). *Let $H(c)$ be a selfadjoint extension of $L(c)$ on $[C_0^\infty(\mathbf{R}^3)]^4$. Then the modified wave operator*

$$W_{\pm}(c)=\text{s-lim}_{t\to\pm\infty}\{\exp(itH(c))\exp(-iX_1(t,c))P_1(c)\\ +\exp(itH(c))\exp(-iX_2(t,c))P_2(c)\}$$

exists and the limit is uniform in $[c_0,\infty)$ *for any positive number* c_0.

The proof of Theorem 1 is similar to Buslaev-Matveev [4] and is given by the stationary phase method. A complete proof of Theorem 1 will be published elsewhere.

Theorem 2. *Under the same assumptions as Theorem* 1 *the strong limit* $W_{\pm}(\infty)=\text{s-lim}_{c\to\infty}W_{\pm}(c)$ *exists and we have*

$$W_{\pm}(\infty)=\begin{bmatrix} w_{\pm}^{(1)} & & & \\ & w_{\pm}^{(1)} & & \\ & & w_{\mp}^{(2)} & \\ & & & w_{\mp}^{(2)} \end{bmatrix},$$

where

$$w_{\pm}^{(j)}=\underset{t\to\pm\infty}{\text{s-lim}}\exp\left\{it\left(-\frac{\Delta}{2}+\tau_jV\right)\right\}\exp\left\{-it\left(-\frac{\Delta}{2}+\frac{\tau_j}{t}z_j^{(n)}(t,D)\right)\right\}$$

$$z_j^{(k)}(t,\xi)=\int_{\operatorname{sgn} t}^{t}V(s\xi+\tau_j\operatorname{grad}_{\xi}z_j^{(k-1)}(s,\xi))ds$$

$$z_j^{(0)}(t,\xi)=0,\qquad n=[1/\delta].$$

Outline of the proof of Theorem 2. The existence of $w_{\pm}^{(j)}$ follows from Alsholm [1] or Buslaev-Matveev [4]. We shall prove only

(3) $$\text{s-lim}_{c\to\infty}W_{+}(c)P_1(c)=W_{+}(\infty)P_1(\infty),$$

where we put

$$P_1(\infty)=\underset{c\to\infty}{\text{s-lim}}P_1(c)=\frac{1}{2}(I+\beta)=\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0\end{bmatrix},$$

since other cases are proved similarly. Set

$$U(t,c)=\exp(itH(c))\exp(-iX_1(t,c))P_1(c)$$

$$U_0(t,c)=\exp\left\{it\left(-\frac{\Delta}{2}+V\right)\right\}\exp\left\{-it\left(-\frac{\Delta}{2}+\frac{1}{t}z_1^{(n)}(t,D)\right)\right\}$$

and consider

(4) $$W_{+}(c)P_1(c)-W_{+}(\infty)P_1(\infty)\\ =(W_{+}(c)P_1(c)-U(t,c))+(U(t,c)-U_0(t)P_1(\infty))\\ +(U_0(t)P_1(\infty)-W_{+}(\infty)P_1(\infty)).$$

Since $\text{s-lim}_{t\to\infty}U(t,c)=W_{+}(c)P_1(c)$ uniformly for $c\geq 1$ by virtue of Theorem 1 and $\text{s-lim}_{t\to\infty}U_0(t)P_1(\infty)=W_{+}(\infty)P_1(\infty)$, we have only to prove

(5) $$\text{s-lim}_{c\to\infty}U(t,c)=U_0(t)P_1(\infty)$$

for any fixed $t>0$ in view of (4). Cirincione-Chernoff [5] gives

(6) $$e^{-itc^2}\exp(itH(c))P_1(c)\longrightarrow\exp\left\{it\left(-\frac{\Delta}{2}+V\right)\right\}P_1(\infty)$$

strongly as $c\to\infty$. On the other hand we have

$$
\begin{aligned}
&e^{itc^2}(\exp(-iX_1(t,c))P_1(c)f)^{\wedge}(\xi)\\
&\quad=e^{itc^2}\exp(-itc\sqrt{|\xi|^2+c^2}-iZ_1^{(n)}(t,c,\xi))P_1(c,\xi)\hat{f}(\xi)\\
&\quad=\exp\left\{-it\frac{|\xi|^2}{1+\sqrt{|\xi|^2/c^2+1}}-iZ_1^{(n)}(t,c,\xi)\right\}P_1(c,\xi)\hat{f}(\xi)\\
&\qquad\longrightarrow\exp(-it|\xi|^2-iz_1^{(n)}(t,\xi))P_1(\infty)\hat{f}(\xi)\qquad(c\to\infty)
\end{aligned}
$$

in $\mathcal{L}^2$, which yields

$$
e^{itc^2}\exp(-iX_1(t,c))P_1(c)\longrightarrow\exp\left\{-it\left(-\frac{\Delta}{2}+\frac{1}{t}z_1^{(n)}(t,D)\right\}P_1(\infty)\tag{7}
$$

strongly as $c\to\infty$. The above (6), (7) and the uniform boundedness of $e^{-itc^2}\exp(itH(c))$ give

$$
\begin{aligned}
U(t,c)&=e^{-itc^2}\exp(itH(c))P_1(c)e^{itc^2}\exp(-iX_1(t,c))P_1(c)\\
&\quad\cdot\exp\left\{it\left(-\frac{\Delta}{2}+V\right)\right\}P_1(\infty)\exp\left\{-it\left(-\frac{\Delta}{2}+\frac{1}{t}z_1^{(n)}(t,D)\right\}P_1(\infty)\\
&=U_0(t)P_1(\infty),\qquad\text{as}\quad c\to\infty,
\end{aligned}
$$

strongly in $\mathcal{L}^2$, which shows (5). Thus the proof is completed.

Remark 3. In Yajima [9] the nonrelativistic limit of the wave operator $W_{\pm}^0(c)$ and the scattering operator is discussed.

Remark 4. The Coulomb potential $V(x)=e/r$ satisfies the condition (A) with $\delta=1$. Then the wave operator $W_{\pm}^0(c)$ does not exist.

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