18. On Nonlinear Hyperbolic Evolution Equations with Unilateral Conditions Dependent on Time

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1. Introduction. In this paper we are concerned with the strong solution of the following nonlinear hyperbolic evolution equation

$$\text{(E)} \qquad \frac{d^2u}{dt^2}(t)+Au(t)+\partial I_{K(t)}\left(\frac{du}{dt}(t)\right)\ni f(t), \qquad 0\leqq t\leqq T$$

in a real Hilbert space H. Here A is a positive self-adjoint operator in H. For each $t\in[0,T]$, $K(t)$ is a closed convex subset of H and $\partial I_{K(t)}$ is the subdifferential of $I_{K(t)}$ which is the indicator function of $K(t)$. We denote the inner product and the norm in H by $(\cdot,\cdot)$ and $|\cdot|$, respectively. For each $t\in[0,T]$, let $P(t)$ denote the projection operator of H onto $K(t)$. Moreover we assume the following conditions for A and $K(t)$.

(A.1) There exists $a\in L^2(0,T;H)$ such that for a.e. $t\in[0,T]$, every $x\in K(t)$ and $\varepsilon>0$, $(1+\varepsilon A)^{-1}(x+\varepsilon a(t))\in K(t)$.

(A.2) There exists a strongly absolutely continuous function $b:[0,T]\to H$ such that $b(t)\in D(A^{1/2})\cap K(t)$ for a.e. $t\in[0,T]$ and $A^{1/2}b\in L^1(0,T;H)$.

(A.3) For each $x\in H$, $P(\cdot)x:[0,T]\to H$ is strongly measurable.

(A.4) There exists a continuous function $\omega:R^+\to R^+$ such that for each $h\in\,]0,T[$ and $v\in C([0,T];H)$,

$$\int_0^{T-h}|P(s+h)v(s)-P(s)v(s)|^2\,ds\leqq h^2\omega\left(\sup_{t\in[0,T]}|v(t)|\right).$$

Definition. Let $u:[0,T]\to H$. Then u is called a strong solution of (E) on $[0,T]$ if (i) $u\in C^1([0,T];H)$, (ii) du/dt is strongly absolutely continuous on $[0,T]$, (iii) $u(t)\in D(A)$ and $du(t)/dt\in K(t)$ for a.e. $t\in[0,T]$ and (iv) u satisfies (E) for a.e. $t\in[0,T]$.

Now we state our main theorem.

Theorem. *Suppose that the assumptions stated above are satisfied. Then for each $f\in W^{1,2}(0,T;H)$, $u_0\in D(A)$ and $v_0\in D(A^{1/2})\cap K(0)$, the equation* (E) *has a unique strong solution u on $[0,T]$ with $u(0)=u_0$ and $(du/dt)(0)=v_0$. Moreover, u has the following properties.*

(i) $Au\in L^\infty(0,T;H)$.

(ii) $u(t)\in D(A^{1/2})$ *for every* $t\in[0,T]$ *and* $A^{1/2}u\in C([0,T];H)$.

(iii) $du(t)/dt\in D(A^{1/2})$ *for* a.e. $t\in[0,T]$ *and* $A^{1/2}du/dt\in L^\infty(0,T;H)$.

(vi) $d^2u/dt^2\in L^2(0,T;H)$.

In the case where $K(t)=K$ is independent of t, the existence and uniqueness of the strong solution of (E) are treated by H. Brézis [3] and the regularity by V. Barbu [1]. These results can be found in V. Barbu [2]. We quoted the assumptions (A.1)–(A.4) from H. Brézis [4].

2. The outline of the proof. The proof of the uniqueness is not difficult and therefore we shall omit it.

To prove the existence, we consider the approximate equations

$$(1)\qquad \begin{cases} \dfrac{d^2u_{\varepsilon,\lambda}}{dt^2}(t)+A_\varepsilon u_{\varepsilon,\lambda}(t)+B_\lambda^t\dfrac{du_{\varepsilon,\lambda}}{dt}(t)=f(t), & 0\leqq t\leqq T \\ u_{\varepsilon,\lambda}(0)=u_0, \quad \dfrac{du_{\varepsilon,\lambda}}{dt}(0)=v_0, \end{cases}$$

for $\varepsilon,\lambda>0$, where $A_\varepsilon=A(1+\varepsilon A)^{-1}$ and $B_\lambda^t=\lambda^{-1}(1-P(t))$. For the solution $u_{\varepsilon,\lambda}$ of (1), we have the following lemma.

Lemma 1. (i) $|u_{\varepsilon,\lambda}(t)|\leqq C_1$. (ii) $\left|\dfrac{du_{\varepsilon,\lambda}}{dt}(t)\right|\leqq C_2$.

(iii) $\displaystyle\int_0^t\left(B_\lambda^s\frac{du_{\varepsilon,\lambda}}{ds},\frac{d^2u_{\varepsilon,\lambda}}{ds^2}\right)ds\geqq -C_3\left(\int_0^t\left|B_\lambda^s\frac{du_{\varepsilon,\lambda}}{ds}\right|^2ds\right)^{1/2}$, *for any* $t\in[0,T]$.

(iv) $\left\|B_\lambda^{\cdot}\dfrac{du_{\varepsilon,\lambda}}{dt}\right\|_{L^2(0,T;H)}\leqq C_4\left(1+\dfrac{1}{\varepsilon}\right)$.

Here C_1, C_2, C_3 and C_4 are positive constants independent of ε, λ and t.

The outline of the proof of Lemma 1 is as follows. We set $du_{\varepsilon,\lambda}/dt=v_{\varepsilon,\lambda}$. (i) and (ii) can be shown by calculating

$$2^{-1}\frac{d}{dt}\{|v_{\varepsilon,\lambda}-b|^2+|A_\varepsilon^{1/2}u_{\varepsilon,\lambda}|^2\}.$$

We can obtain (iii) noticing

$$\begin{aligned}
&(2\lambda)^{-1}|(1-P(s+h))v_{\varepsilon,\lambda}(s+h)|^2-(2\lambda)^{-1}|(1-P(s))v_{\varepsilon,\lambda}(s)|^2\\
&\quad-(\lambda^{-1}(1-P(s))v_{\varepsilon,\lambda}(s),\ v_{\varepsilon,\lambda}(s+h)-v_{\varepsilon,\lambda}(s))\\
&\quad=I+II+III,\\
&I=(2\lambda)^{-1}|(1-P(s+h))v_{\varepsilon,\lambda}(s+h)|^2-(2\lambda)^{-1}|(1-P(s+h))v_{\varepsilon,\lambda}(s)|^2\\
&\quad-(\lambda^{-1}(1-P(s+h))v_{\varepsilon,\lambda}(s),\ v_{\varepsilon,\lambda}(s+h)-v_{\varepsilon,\lambda}(s)),\\
&II=(2\lambda)^{-1}|(1-P(s+h))v_{\varepsilon,\lambda}(s)|^2-(2\lambda)^{-1}|(1-P(s))v_{\varepsilon,\lambda}(s)|^2,\\
&III=-\lambda^{-1}((P(s+h)-P(s))v_{\varepsilon,\lambda}(s),\ v_{\varepsilon,\lambda}(s+h)-v_{\varepsilon,\lambda}(s)),\\
&I\leqq\lambda^{-1}|v_{\varepsilon,\lambda}(s+h)-v_{\varepsilon,\lambda}(s)|^2\leqq\frac{1}{\lambda}h^2\sup_{t\in[0,T]}\left|\frac{dv_{\varepsilon,\lambda}}{dt}(t)\right|^2,\\
&II\leqq|P(s+h)v_{\varepsilon,\lambda}(s)-P(s)v_{\varepsilon,\lambda}(s)||B_\lambda^s v_{\varepsilon,\lambda}(s)|\\
&\quad+(2\lambda)^{-1}|P(s+h)v_{\varepsilon,\lambda}(s)-P(s)v_{\varepsilon,\lambda}(s)|^2,\\
&III\leqq\lambda^{-1}|P(s+h)v_{\varepsilon,\lambda}(s)-P(s)v_{\varepsilon,\lambda}(s)||v_{\varepsilon,\lambda}(s+h)-v_{\varepsilon,\lambda}(s)|,
\end{aligned}$$

and the assumption (A.4). (iv) can be obtained by multiplying the first equation of (1) by $B_\lambda^t du_{\varepsilon,\lambda}/dt$ and integrating over $[0,T]$.

Let $\varepsilon>0$ be fixed. By the same manner as in Theorem 3.1 of H. Brézis [5], it follows from Lemma 1 (iv) that $\lim_{\lambda\to0}u_{\varepsilon,\lambda}=u_\varepsilon$ and $\lim_{\lambda\to0}du_{\varepsilon,\lambda}/dt=du_\varepsilon/dt$ exist in $C([0,T];H)$ and u_ε is the strong solution of the equation

$$(2)\qquad \begin{cases} \dfrac{d^2u_\varepsilon}{dt^2}(t)+A_\varepsilon u_\varepsilon(t)+\partial I_{K(t)}\Big(\dfrac{du_\varepsilon}{dt}(t)\Big)\ni f(t), & 0\leqq t\leqq T \\ u_\varepsilon(0)=u_0, \quad \dfrac{du_\varepsilon}{dt}(0)=v_0. \end{cases}$$

Letting $\lambda\to 0$ in Lemma 1 (iii) and using that u_ε is the solution of (2) we obtain

$$(3)\qquad \int_0^t \left|\frac{d^2u_\varepsilon}{ds^2}\right|^2 ds\leqq M\Big(1+\int_0^t |A_\varepsilon u_\varepsilon|^2\, ds\Big) \qquad \text{for any } t\in[0,T],$$

where M is a constant independent of ε and t. From (3), the assumption (A.1) and the definition of $\partial I_{K(t)}$, we get the following lemma.

Lemma 2. (i) $|A_\varepsilon u_\varepsilon(t)|\leqq C_5$. (ii) $\varepsilon^{1/2}\left|A_\varepsilon \dfrac{du_\varepsilon}{dt}(t)\right|\leqq C_6$.

(iii) $\left|A^{1/2}(1+\varepsilon A)^{-1}\dfrac{du_\varepsilon}{dt}(t)\right|\leqq C_7$. (iv) $\left\|\dfrac{d^2u_\varepsilon}{dt^2}\right\|_{L^2(0,T;H)}\leqq C_8$.

Here C_5, C_6, C_7 and C_8 are constants independent of ε and t.

If $\varepsilon,\delta>0$, then by using (2) and the monotonicity of $\partial I_{K(t)}$ we have for a.e. $s\in[0,T]$

$$(4)\qquad \begin{aligned} &\frac{1}{2}\frac{d}{ds}\Big\{\Big|\frac{du_\varepsilon}{ds}(s)-\frac{du_\delta}{ds}(s)\Big|^2 \\ &\qquad\qquad +|A^{1/2}(1+\varepsilon A)^{-1}u_\varepsilon(s)-A^{1/2}(1+\delta A)^{-1}u_\delta(s)|^2\Big\} \\ &\leqq \Big(A_\varepsilon u_\varepsilon(s)-A_\delta u_\delta(s),\ \varepsilon A_\varepsilon\frac{du_\varepsilon}{ds}(s)-\delta A_\delta\frac{du_\delta}{ds}(s)\Big) \\ &\leqq (|A_\varepsilon u_\varepsilon(s)|+|A_\delta u_\delta(s)|)\Big(\varepsilon\Big|A_\varepsilon\frac{du_\varepsilon}{ds}(s)\Big|+\delta\Big|A_\delta\frac{du_\delta}{ds}(s)\Big|\Big). \end{aligned}$$

Integrating (4) over $[0,T]$ and using Lemma 2 (i) and (ii), it follows that $\lim_{\varepsilon\to 0} u_\varepsilon=u$ and $\lim_{\varepsilon\to 0} du_\varepsilon/dt=du/dt$ exist in $C([0,T];H)$. By the standard theory of maximal monotone operators, we can prove that u is the strong solution of (E) and satisfies the properties (i)–(iv) of Theorem.

3. Example. Let Ω be a bounded domain in R^n having a sufficiently smooth boundary Γ. We set $Q=]0,T[\times\Omega$ and $\Sigma=]0,T[\times\Gamma$. Let $\psi\in L^2(0,T;H)$ be such that $\partial\psi/\partial t\in L^2(Q)$ and $\psi(t,x)\leqq 0$ a.e. on Σ. Consider the following hyperbolic unilateral problem:

$$(\mathrm{U})\qquad \begin{aligned} &\frac{\partial u}{\partial t}\geqq\psi && \text{a.e. on } Q, \\ &\frac{\partial^2 u}{\partial t^2}=\Delta u+f && \text{a.e. on } \Big\{(t,x)\in Q\,;\,\frac{\partial u}{\partial t}>\psi\Big\}, \\ &\frac{\partial^2 u}{\partial t^2}\geqq\Delta u+f && \text{a.e. on } \Big\{(t,x)\in Q\,;\,\frac{\partial u}{\partial t}=\psi\Big\}, \\ &u(t,x)=0 && \text{a.e. on } \Sigma, \end{aligned}$$

$$u(0,x)=u_0(x); \quad \frac{\partial u}{\partial t}(0,x)=v_0(x) \qquad \text{a.e. on } \Omega.$$

Corollary. *Let u_0, v_0 and f be given satisfying:*

$$u_0 \in H_0^1(\Omega)\cap H^2(\Omega), \quad v_0 \in H_0^1(\Omega), \quad v_0(x)\geqq \psi(0,x) \qquad \text{a.e. } on\ \Omega.$$

$$f,\ \frac{\partial f}{\partial t}\in L^2(Q).$$

Then problem (U) *has a unique solution u which satisfies:*

$$u \in C([0,T];H_0^1(\Omega))\cap L^\infty(0,T;H^2(\Omega)),$$

$$\frac{\partial u}{\partial t}\in C([0,T];L^2(\Omega))\cap L^\infty(0,T;H_0^1(\Omega)),$$

$$\frac{\partial^2 u}{\partial t^2}\in L^2(0,T;L^2(\Omega)).$$

Proof of Corollary. We take $H=L^2(\Omega)$, $Av=-\Delta v$ for $v\in D(A)=H_0^1(\Omega)\cap H^2(\Omega)$ and

$$K(t)=\{v\in L^2(\Omega);\ v(x)\geqq\psi(t,x) \text{ a.e. on } \Omega\}.$$

Taking $a(t)=-\Delta\psi(t,x)$ and $b(t)=\max\{0,\psi(t,x)\}$ the assumption (A.1) and (A.2) is realized, respectively. Since $P(t)v(x)=\max\{v(x),\psi(t,x)\}$, the assumption (A.3) is satisfied. The assumption (A.4) is realized taking $\omega=\|\partial\psi/\partial t\|_{L^2(0,T;H)}$ (constant). Therefore we can apply Theorem and we know that the equation (E) has a unique strong solution u. By the same manner as in Corollary 3.4 in Chapter IV of V. Barbu [2], it follows that u satisfies (U) in the generalized sense.

References

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