16. *Transmutation Theory for Certain Radial Operators*

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1. **Introduction.** For $Q_0u=(\Delta_Q u')'/\Delta_Q$ based on a radial Laplace-Beltrami operator we studied general transmutation theory for operators $\hat{Q}u=Q_0u+\hat{q}(x)u$ in [4] using various solutions of $\hat{Q}u=-k^2u$ as important ingredients. In the present work we consider operators $\tilde{Q}u=x^2Q_0u+x^2\{k^2-\tilde{q}(x)\}u$ (with corresponding eigenfunction equations $\tilde{Q}u=\lambda^2u$) and will concentrate on the case $\Delta_Q=x^2$ which arises in various scattering problems (cf. [1], [2], [6]–[9], [11], [12]) so a certain amount of guideline information is available (cf. also [5]-B transmutes P into $Q, B: P\to Q$, if $QBf=BPf$ for suitable f). We show here that with suitable modifications most of the constructions and techniques of the $\hat{Q}$ theory have a version in the $\tilde{Q}$ theory and we describe some of the basic transmutations and connection formulas.

2. **Basic constructions.** We take $\Delta_Q=x^2$ and set $\varphi=xu$ so $\tilde{Q}u=\lambda^2u$ is

(2.1) $$x^2\varphi''+x^2\{k^2-\tilde{q}(x)\}\varphi=\lambda^2\varphi$$

($\tilde{q}$ real) and one writes $\lambda^2=\sigma(\sigma+1)=\nu^2-1/4$ (so $\sigma\sim l=$angular momentum and $\nu\sim l+1/2$). We denote by $\varphi(\nu,k,x)$ the "regular solution" of (2.1) ($\varphi\sim x^{\nu+1/2}$ as $x\to 0$) and by $f(\nu,\pm k,x)$ the "Jost solutions" (e.g. $f(\nu,-k,x)\sim e^{ikx}$ as $x\to\infty$) with the "Jost function" $f(\nu,-k)=W(f(\nu,-k,x),\varphi(\nu,k,x))$ ($W(f,g)=fg'-f'g$). Assume e.g. $\int_0^\infty x|\tilde{q}|\,dx<\infty$ and $\int_0^\infty x^2|\tilde{q}|\,dx<\infty$ as in [2] but we do not emphasize hypotheses on $\tilde{q}$ (cf. [1], [6], [9], [11], [12]); we want mainly $\varphi\sim\varphi_0$ and $f\sim f_0$ as e.g. $|\nu|\to\infty$, $\mathrm{Re}\,\nu>0$, where (corresponding to $\tilde{q}=0$)

(2.2) $$\varphi_0(\nu,k,x)=2^\nu\Gamma(\nu+1)k^{-\nu}x^{1/2}J_\nu(kx);$$
$$f_0=((1/2)\pi kx)^{1/2}e^{(1/2)i\pi(\nu+1/2)}H^1_\nu(kx);$$

($f_0=f_0(\nu,-k,x)$) and

$$f_0(\nu,-k)=2^\nu(2/\pi)^{1/2}\Gamma(\nu+1)k^{-\nu+1/2}\exp\{(1/2)i\pi(\nu-1/2)\}.$$

We think of k as fixed here and one knows then that $f(\nu,-k,x)$ is entire in ν while $\varphi(\nu,k,x)$ and $f(\nu,-k)$ are analytic for $\mathrm{Re}\,\nu>0$ (the range of analyticity can be enlarged with suitable hypotheses on $\tilde{q}$). We follow formally now the procedure in [2] with some refinements and elaboration. Thus set $g(\nu,-k,r)=f(\nu,-k,r)/r$ and let Z denote the zeros ν_j (if any) of $f(\nu,-k)$ in $\mathrm{Re}\,\nu>0$ with

$$M^2(\nu_j, k)=\int_0^\infty g^2(\nu_j, -k, r)dr.$$

Such ν_j are simple zeros and one sets $d\rho(\nu)=\sum \delta(\nu-\nu_j)/M^2(\nu_j, k)$ for $\nu\in Z$ with $d\rho(\nu)=2i\nu^2 d\nu/\pi f(\nu, -k)f(-\nu, -k)$ for $\nu\in[0, i\infty)$. From [2] one has the formal completeness relation

$$\delta(r-s)=\langle g(\nu, -k, r), g(\nu, -k, s)\rangle_\rho\sim\int g(\nu, -k, r)g(\nu, -k, s)d\rho(\nu)$$

and we show then (g_1, ρ^1, etc. refer to an operator $\tilde{Q}_1$ based on potential $\tilde{q}_1$).

Theorem 2.1. *Define* $\beta(r, s)=\langle g(\nu, -k, r), g_1(\nu, -k, s)\rangle_\rho$ *and* $\tilde{\beta}(r, s)=\langle g(\nu, -k, r), g_1(\nu, -k, s)\rangle_{\rho^1}$ *with* $\tilde{\mathscr{B}}f(s)=\langle\beta(r, s), f(r)\rangle$ *and* $\tilde{B}f(r)=\langle\tilde{\beta}(r, s), f(s)\rangle$ *for suitable* f. *The* r *and* s *brackets refer to distribution pairings on* $[0, \infty)$ *and one has triangularity* $\beta(r, s)=0$ *for* $s>r$ *with* $\tilde{\beta}(r, s)=0$ *for* $r>s$. *Set* $\mathfrak{G}f(\nu)=\hat{f}(\nu)=\int_0^\infty f(s)g(\nu, -k, s)ds$ *so that formally* $G\hat{f}(r)=\mathfrak{G}^{-1}\hat{f}(r)=f(r)=\langle\hat{f}(\nu), g(\nu, -k, r)\rangle_\rho$. *Then* $\tilde{B}:\tilde{Q}_1\to\tilde{Q}$ *and* $\tilde{\mathscr{B}}(\sim\tilde{B}^{-1}):\tilde{Q}\to\tilde{Q}_1$ *are transmutations with* $\tilde{\mathscr{B}}\{g(\nu, -k, \cdot)\}(r)=g_1(\nu, -k, r)$ *and* $\tilde{B}\{g_1(\nu, -k, \cdot)\}(r)=g(\nu, -k, r)$. *Set* $B=\tilde{\mathscr{B}}^*$ (so $Bf(r)=\langle\beta(r, s), f(s)\rangle$) *and correspondingly* $\mathscr{B}=\tilde{B}^*$; *then* $\mathfrak{G}Bf=\mathfrak{G}_1 f$ *and* $\mathfrak{G}f=\mathfrak{G}_1\mathscr{B}f$ (with $\mathscr{B}=B^{-1}$).

We indicate next a connection to an exterior transmutation developed in [7], [8]. Thus for Q_0 based on $\Delta_Q=x^{n-1}$ one considers $\tilde{Q}=x^2Q_0+x^2\{k^2-\tilde{q}(x)\}$ and $\tilde{P}=x^2Q_0+x^2k^2$. For suitable $\tilde{q}$ a kernel $K(r, s)$ is constructed in [7], [8] (by successive approximations) such that the formula

$$(2.3)\qquad u(r, \cdot)=\{B_e h\}(r, \cdot)=h(r, \cdot)+\int_r^\infty s^{n-3}K(r, s)h(s, \cdot)ds$$

links suitable solutions h of $(\Delta_n+k^2)h=0$ to corresponding solutions u of $\{\Delta_n+(k^2-\tilde{q}(r))\}u=0$. The kernel $K(r, s)$ satisfies $\tilde{Q}_r K=\tilde{P}_s K$ for $s>r$ with $2r^{n-2}K(r, r)=\int_r^\infty s\tilde{q}(s)ds$. If we write $\check{K}(r, s)=K(r, s)Y(s-r)$ (Y the Heaviside function) then one can show

Theorem 2.2. *For suitable* $\tilde{q}$ *the map* $B_e f(r)=f(r)+\langle\check{K}(r, s), f(s)\rangle$ *is a transmutation* $\tilde{P}\to\tilde{Q}$ *and for* $n=3$, $\delta(s-r)+\check{K}(r, s)\sim\tilde{\beta}(r, s)=\langle g(\nu, -k, r), g_1(\nu, -k, s)\rangle_{\rho^1}$ (where $\tilde{Q}_1\sim\tilde{P}$ and ρ^1 is the "free" measure indicated below).

Example 2.3. We denote by "free" the case where $\tilde{q}=0$ so that (2.2) holds. In this event $f(\nu, -k)$ has no zeros for $\mathrm{Re}\,\nu>0$ and $d\rho(\nu)=-(\nu/\pi k)\sin\pi\nu d\nu$ is the "free" measure. The inversion theory for $\mathfrak{G}$ is the Kontorovič-Lebedev theory which can be treated in various forms (cf. [10]). The version which we obtain below (cf. (2.5)) specializes for $\tilde{q}=0$ to

$$(2.4)\qquad \tilde{G}(\nu)=\int_0^\infty G(s)H_\nu^1(ks)ds;\qquad rG(r)=\frac{1}{2}\int_{-i\infty}^{i\infty}\nu\tilde{G}(\nu)J_\nu(kr)d\nu.$$

In order to arrive at a general form of (2.4) we suppose $f(\nu, -k)$ has no zeros for $\mathrm{Re}\,\nu>0$ so that $d\rho(\nu)=\hat{\rho}(\nu)d\nu$. From properties of $f(\pm\nu, -k, x)$ and $\varphi(\pm\nu, k, x)$ one has

$$rf(r)=\langle \hat{f}(\nu), f(\nu, -k, r)\rangle_{\rho}=\frac{1}{2}\int_{-i\infty}^{i\infty} \hat{f}(\nu)f(\nu, -k, r)\hat{\rho}(\nu)d\nu$$

and from this

$$rf(r)=-(i/\pi)\int_{-i\infty}^{i\infty} \nu\hat{f}(\nu)\Phi(\nu, k, r)d\nu \tag{2.5}$$

where $\Phi(\nu, k, r)=\varphi(\nu, k, r)/f(\nu, -k)$ (cf. [4]) and using the formal relation (*) $-(i\mu/\pi)\int_{0}^{\infty} \Phi(\mu, k, s)g(\nu, -k, s)ds/s=\delta(\mu-\nu)$ arising from (2.5) we show

Theorem 2.4. *Given absolutely continuous $d\rho(\nu)=\hat{\rho}(\nu)d\nu$ the inversion (2.5) holds. If $\tilde{Q}$ and $\tilde{Q}_1$ both have continuous spectrum then B is characterized by $B\{g_1(\nu, -k, \cdot)\}(r)=(\hat{\rho}/\hat{\rho}_1)(\nu)g(\nu, -k, r)$ and in addition*

$$B\{\Phi_1(\nu, k, s)/s\}(r)=\langle \beta(r, s), \Phi_1(\nu, k, s)/s\rangle=\Phi(\nu, k, r)/r. \tag{2.6}$$

One can construct a formal proof of (2.6) following [4] (using analytic continuation) but a simpler formal verification can be obtained by looking at $\langle \beta(r, s), g(\nu, -k, r)\rangle=g_1(\nu, -k, s)$ as an extension of $\mathfrak{G}$ to β, so that $\hat{\beta}(\nu, s)=g_1(\nu, -k, s)$, using the inversion (2.5), and then applying (*) for Φ_1 and g_1.

3. General techniques. First, assuming $g(\nu, -k, 1)=0$ on the spectrum,

$$U(r, s)=\langle \hat{f}(\nu)/g(\nu, -k, 1),\ g(\nu, -k, r)g(\nu, -k, s)\rangle_{\rho} \tag{3.1}$$

where $\hat{f}(\nu)=\mathfrak{G}f(\nu)$ makes sense formally and using the idea of generalized translation developed by Hutson-Pym (cf. [3], [4]) one has for suitable $\tilde{q}$

Theorem 3.1. *$U(r, s)=T_s^r f(s)$ represents a generalized translation for $\tilde{Q}$ determined by $\tilde{Q}_r U=\tilde{Q}_s U$, $U(1, s)=f(s)$, and $D_r U(1, s)=Cf(s)$ $=\langle \Gamma(s, \eta), f(\eta)\rangle$ where*

$$\Gamma(s, \eta)=\langle g(\nu, -k, s)g(\nu, -k, \eta),\ Dg(\nu, -k, 1)/g(\nu, -k, 1)\rangle_{\rho}$$

($\tilde{Q}C=C\tilde{Q}$ and $Cf(1)=f'(1)$).

The "Cauchy problem" indicated in Theorem 3.1 is to be considered in two regions $r, s\geq 1$ and $0\leq r$, $s\leq 1$. It can be transformed into two halfplane Cauchy problems $\eta\geq 0$ and $\eta\leq 0$ respectively by setting $\eta=\log r$ and $\xi=\log s$, from which standard uniqueness results can be transported; the "data" is given on $-\infty<\xi<\infty$.

Theorem 3.2. *Let $\tilde{Q}$ and $\tilde{Q}_1$ be based on $\Delta_Q=x^{n-1}$ as above and let A and C be linear operators commuting with $\tilde{Q}_1$. Let φ be the unique solution of $\tilde{Q}_r\varphi=\tilde{Q}_s^1\varphi$ ($\tilde{Q}^1\sim\tilde{Q}_1$), $\varphi(1, s)=Af(s)$, and $D_r\varphi(1, s)=Cf(s)$. Then $Bf(r)=\varphi(r, 1)$ determines a transmutation $B: \tilde{Q}_1\rightarrow\tilde{Q}$.*

Remark 3.3. In this spirit one can formally construct B and $\breve{B}$

via Cauchy type problems as follows ($n=3$). Let $U_1(t,s)$ have the form (3.1) with $(\rho, g, \mathfrak{G})$ replaced by $(\rho^1, g_1, \mathfrak{G}_1)$ etc. Set $\varphi(r,s) = \langle \beta(r,t), U_1(t,s)\rangle$ and $\tilde{\varphi}(r,s)=\langle \tilde{\beta}(r,t), U_1(t,s)\rangle$ so that $\varphi(r,1)=Bf(r)$ and $\tilde{\varphi}(r,1)=\tilde{B}f(r)$. For suitable f we obtain e.g. $\tilde{\varphi}(1,s)=\tilde{A}f(s) = \langle f(\sigma), \tilde{\mathfrak{A}}(s,\sigma)\rangle$, $D_r\tilde{\varphi}(1,s)=\tilde{C}f(s)=\langle f(\sigma), \tilde{\mathfrak{C}}(s,\sigma)\rangle$, where formally $\tilde{\mathfrak{A}}(s,\sigma) = \langle \alpha(\nu,k)g_1(\nu,-k,s), g_1(\nu,-k,\sigma)\rangle_{\rho^1}$ and $\tilde{\mathfrak{C}}(s,\sigma)=\langle \gamma(\nu,k)g_1(\nu,-k,s), g_1(\nu,-k,\sigma)\rangle_{\rho^1}$ $(\alpha(\nu,k)=g(\nu,-k,1)/g_1(\nu,-k,1)$ and $\gamma(\nu,k)=Dg(\nu,-k,1)/g_1(\nu,-k,1))$. Similar formulas apply for $\varphi(1,s)$ and $D_r\varphi(1,s)$ with ρ^1 replaced by ρ in the corresponding $\mathfrak{A}(s,\sigma)$ and $\mathfrak{C}(s,\sigma)$.

By modifying some techniques in [4] one shows (cf. also [3])

Theorem 3.4. *For suitable f, h and T_s^r defined as in Theorem 3.1 there results $\langle T_s^r f(s), h(s)\rangle = \langle f(s), T_s^r h(s)\rangle$ and setting $(f*h)(r) = \langle T_s^r f(s), h(s)\rangle$ it follows that $(f*h)^\wedge = \hat{f}\hat{h}/g(\nu,-k,1)$.*

Remark 3.5. Following [4] it is possible to develop various Gelfand-Levitan (G-L) equations. For example based on the equations $g(\nu,-k,r)=\langle \tilde{\beta}(r,s), g_1(\nu,-k,s)\rangle$ and $g_1(\nu,-k,t)=\langle \beta(u,t), g(\nu,-k,u)\rangle$ a G-L equation arises in the form $\beta(r,t)=\langle \tilde{\beta}(r,s), A(s,t)\rangle$ where $A(s,t) = \langle g_1(\nu,-k,s), g_1(\nu,-k,t)\rangle_\rho$.

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