142. A Note on the Number of Irreducible Characters in a p-Block with Normal Defect Group

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1. Let G be a finite group and p be a prime. Let B be a p-block of G with defect group D. We denote by $k(B)$ the number of ordinary irreducible characters in B. R. Brauer [1] conjectured

$$(K): k(B) \leqq |D|.$$

In [5] it is shown that (K) is true if G is p-solvable and if p is sufficiently large compared with the sectional rank of D.

The purpose of this note is to prove the following

Theorem. *For any positive integer n, there exists a constant b_n depending only on n such that the following statement is true: Let B be a p-block of a group G with normal defect group D. Assume that the sectional rank of D equals n. Then, if p is larger than b_n, we have $k(B) \leqq |D|$.*

2. Let B be a p-block of a group G with defect group D, which is normal in G. Let b be a p-block of $DC_G(D)$ covered by B and T_b be the inertia group in G of the block b. Then $[T_b : DC_G(D)]$ is prime to p. Let B' be the unique block of T_b that covers b. Then D is the defect group of B' and $k(B')=k(B)$. In order to prove that (K) is true for B we may assume that $G=T_b$, $B=B'$. Then $G/C_G(D)$ contains the normal p-Sylow group $DC_G(D)/C_G(D)$, so that $G/C_G(D)$ has a p-complement $L/C_G(D)$. Set $\bar{L}=L/C_G(D)$. Form the semi-direct product $H=\bar{L}D$ with the natural (faithful) action of $\bar{L}$ on D. Theorem follows immediately from the result in [5] mentioned above and the following

Proposition. *Let the notation be as above. We have $k(B) \leqq \mathrm{cl}(H)$. Here $\mathrm{cl}(X)$ denotes the number of conjugacy classes of X for a group X.*

Proof. Let θ be the canonical character of b. For every irreducible character χ of D, define the class function $\tilde{\chi}$ on $DC_G(D)$ as follows:

$$\tilde{\chi}(z)=\begin{cases}\chi(x)\theta(y) & \text{if } x \in D\\ 0 & \text{otherwise,}\end{cases}$$

where x and y denote the p-part and p'-part of $z \in DC_G(D)$, respectively. Then the map $\sim$ is a bijection from the set of irreducible characters of D onto the set of irreducible characters in b (see [2], (V. 4.7)). Let $\{\chi_i\}$ be a complete set of representatives of $\bar{L}$-conjugate classes of irreducible characters of D. Note that $\{\tilde{\chi}_i\}$ is then a complete set of representatives of G-conjugate classes of irreducible characters in b, since $G=LDC_G(D)$. As an irreducible character of G lies in B if and only if the restriction of it to $DC_G(D)$ contains precisely one of $\tilde{\chi}_i$'s, Clifford's theorem implies that

(1) $$k(B)=\sum_i |\mathrm{Irr}\,(I_G(\tilde{\chi}_i)\,|\,\tilde{\chi}_i)|.$$

Here $\mathrm{Irr}\,(I_G(\tilde{\chi}_i)\,|\,\tilde{\chi}_i)$ denotes the set of irreducible characters of $I_G(\tilde{\chi}_i)$, the inertia group in G of $\tilde{\chi}_i$, whose restriction to $DC_G(D)$ contain $\tilde{\chi}_i$. By Gallagher [4], Theorem,

(2) $$|\mathrm{Irr}\,(I_G(\tilde{\chi}_i)\,|\,\tilde{\chi}_i)|\leqq \mathrm{cl}\,(I_G(\tilde{\chi}_i)/DC_G(D)).$$

Set $I_{\bar{L}}(\chi_i)=\{x\in \bar{L}\,|\,\chi_i^x=\chi_i\}$, for each i. Then it is shown that $I_{\bar{L}}(\chi_i)\cong I_G(\tilde{\chi}_i)/DC_G(D)$. This, together with (1) and (2), implies

(3) $$k(B)\leqq \sum_i \mathrm{cl}\,(I_{\bar{L}}(\chi_i)).$$

Now let B_0 be the principal p-block of H. (Note that B_0 is the unique p-block of H.) Replace, in the above, G, B by H, B_0 respectively. We repeat the same argument as above. In this case the equality holds in (2) by Gallagher [3], Theorem 7. Thus we obtain

(3)′ $$k(B_0)=\sum_i \mathrm{cl}\,(I_{\bar{L}}(\chi_i)).$$

Since $k(B_0)=\mathrm{cl}\,(H)$, (3) and (3)′ complete the proof.

References

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