136. An Integro-Differential Operator and the Associated Semigroup of Operators

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1. As in [4], the generator of a certain type of semigroup is represented as an integro-differential operator.

Here, we solve a converse problem for an operator of this type, which is spatially homogeneous on R^N. Let

(1) $$Af(x)=A_Df(x)+A_If(x), \qquad x=(x_1,\cdots,x_N)\in R^N,$$
$$A_Df(x)=\sum_{|\alpha|\leqslant m} a_\alpha D_\alpha f(x),$$
$$A_If(x)=\int_{R^N\setminus\{0\}}\{f(y+x)-\rho(y)\sum_{|\alpha|<n}1/\alpha!\,D_\alpha f(x)y_\alpha\}\mu(dy).$$

a_α's are complex constants and $D_\alpha f(x)=\partial^{|\alpha|}f(x)/\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}$, $|\alpha|=\alpha_1+\cdots+\alpha_N$, $\alpha!=\alpha_1!\cdots\alpha_N!$ and $y_\alpha=y_1^{\alpha_1}\cdots y_N^{\alpha_N}$ for multi-index $\alpha=(\alpha_1,\cdots,\alpha_N)$. μ is a complex valued σ-finite measure on $(R^N\setminus\{0\},\mathcal{B}_{R^N\setminus\{0\}})$ such that

(2) $$\int_{0<|y|\leqslant 1}|y|^n|\mu|(dy)+|\mu|(1<|y|<\infty)<\infty.$$

$\rho(y)$ is an isotropic C^∞ function on R^N such that

(3) $0<\rho(y)\leqslant 1$, $|y|^{n-1}\rho(y)\leqslant 1$, $y\in R^N$, $1-\rho(y)=\mathcal{O}(|y|^{n+1})$, as $y\to 0$.

We assume that $a_\alpha\neq 0$ for some α with $|\alpha|=m$, except the case $m=0$.

The problem here is to obtain the fundamental solution $Q(t,x,\cdot)$ of

(4) $$(\partial/\partial t)u(t,x)=Au(t,x),$$

when A is essentially of elliptic type.

2. Let

$$a(z)=a_D(z)+a_I(z),$$

where

$$a_D(z)=\sum_{|\alpha|\leqslant m} i^{|\alpha|}a_\alpha z_\alpha,$$

(5) $$a_I(z)=\int_{R^N\setminus\{0\}}\left\{e^{iy\cdot z}-\rho(y)\sum_{k=0}^{n-1}1/k!\,(iy\cdot z)^k\right\}\mu(dy), \qquad y\cdot z=\sum_{j=1}^N y_jz_j.$$

The measure μ is called *degenerate*, if its support is contained in some hyperplane, which passes through the origin and has dimension at most $N-1$. μ is called *rapidly decreasing at* ∞, if, for each natural number l,

(6) $$\int_{|y|>1}|y|^l|\mu|(dy)<\infty.$$

For the second part A_I of A, we have

Theorem 1. *Let μ be given by a positive measure μ_+ as*

(7)[1] $$\mu=(-1)^{[(n-1)/2]}\mu_+.$$

1) For a real number s, $[s]$ denotes the largest integer l such that $l\leqslant s$.

Then, there is a bounded signed measure $Q_I(t,\cdot)$ *on* $(R^N, \mathcal{B}_{R^N})$ *for* $t\geqslant 0$ *such that*

$$e^{ta_I(z)}=\int_{R^N} e^{ix\cdot z}Q_I(t,dx), \qquad z\in R^N.$$

(i) *When* $n\geqslant 3$ *and* μ *is non-degenerate,* $Q_I(t,\cdot)$, $t>0$, *has a density function* $q_I(t,x)$ *with respect to the Lebesgue measure, which is* C^∞ *in* (t,x) *and*

$$q_I(t,x)=\int_{R^N} q_0(t,x-y)Q_\infty(t,dy).$$

$q_0(t,x)$ *is in* $\mathcal{S}=\mathcal{S}(R^N)$ *as a function in* x, *and* $Q_\infty(t,\cdot)$ *is a generalized*[2] *compound Poisson distribution on* R^N:

(8)[3] $\qquad Q_\infty(t,E)=e^{-bt}\sum_{k=0}^{\infty} t^k/k!\,\mu_\infty^{*k}(E), \qquad E\in\mathcal{B}_{R^N},$

$\qquad \mu_\infty(E)=\mu(E\cap\{y\,|\,|y|>r\}),\ b=\mu_\infty(R^N), \qquad$ *for an* $r>0$.

In particular, if μ *is rapidly decreasing at* ∞, *then* $q_I(t,x)$ *itself is in* $\mathcal{S}$.

(ii) *When* $n\geqslant 3$ *and* μ *is degenerate, let* H *be the minimal hyperplane which contains the support of* μ *and the origin* O. *Then,* $Q_I(t,\cdot)$ *is concentrated on* H, *and has a density function* $\tilde{q}_I(t,x)$ *with respect to the Lebesgue measure* $d\tilde{x}$ *on* H. $\tilde{q}_I(t,\tilde{x})$ *has the same properties of* $q_I(t,x)$ *as a function on* H, *and*

$$e^{ta_I(z)}=e^{ta_I(P_H(z))}=\int_H e^{i\tilde{x}\cdot\tilde{P}_H(z)}\tilde{q}_I(t,\tilde{x})d\tilde{x}, \qquad z\in R^N,$$

where $P_H(z)$ *is the orthogonal projection of* z *on* H.

(iii) *When* $n\leqslant 2$,

$$e^{-d_\rho t}Q_I(t,\cdot)$$

is a probability measure on R^N, *more precisely, an infinitely devisible distribution on* R^N, *where*

$$d\rho=\int_{R^N\setminus\{0\}}(1-\rho(y))\mu(dy).$$

3. Let $B=B(R^N)$ be the set of all complex valued bounded measurable function on R^N, and let $L^1=L^1(R^N,dx)$. $C_0=C_0(R^N)$ denotes the set of all continuous functions such that $f(x)\to 0$, as $|x|\to\infty$. For $l=0,1,2,\cdots$, let $M_l=M_l(R^N)$ be the set of all complex valued bounded measures ν on $(R^N,\mathcal{B}_{R^N})$ such that

$$\int_{R^N}(1+|x|^l)\,|\nu|\,(dx)<\infty.$$

F_l denotes the set of all Fourier transforms $\mathcal{F}\nu$ of ν in M_l:

$$\mathcal{F}\nu(z)=\int_{R^N} e^{ix\cdot z}\nu(dx).$$

We write $|||f|||=|||\mathcal{F}^{-1}f|||$ for the total variation of $\mathcal{F}^{-1}f$ for f in F_0.

2) $\mathcal{S}(R^N)$ denotes the set of all rapidly decreasing functions on R^N in the sense of L. Schwartz.

3) For a complex valued bounded measure ν, ν^{*k} is the k-th power with respect to the convolution.

Theorem 2. *Let μ be given by a positive measure μ_+ as in* (7), *and let*

(9) $$\mathcal{R}e\,(a_D(z)) \leqslant c_0 < \infty, \qquad z \in R^N.$$

(i) *Then, in each of*

Case 1. *A_D is elliptic in the following sense:*

(10) $$\mathcal{R}e\,(a_D(z)) \leqslant -c_D|z|^2 + d_D, \qquad z \in R^N.$$

Case 2. *$n \geqslant 3$ and μ is non-degenerate,*

Case 3. *$m \leqslant 1$, there is a complex valued bounded measure $Q(t,\cdot)$ for $t \geqslant 0$, such that*

$$e^{ta(z)} = \int_{R^N} e^{ix\cdot z} Q(t, dx), \qquad z \in R^N.$$

(ii) *In Cases* 1–3, *let*

$$Q_t f(x) = \int_{R^N} f(y+x) Q(t, dy),$$

when the integral converges for a measurable function f.

Then, $\{Q_t, t \geqslant 0\}$ is a semigroup of operators on each of B, C_0 and F_l, $l = 0, 1, 2, \cdots$, respectively.

$$u(t,x) = Q_t f(x), \quad t > 0, \quad x \in R^N,$$

satisfies (4) *for each f in $F_{m\vee n}$, and*

(11) $$\lim_{t\to 0} u(t,x) = f(x), \qquad f \in F_0,$$

with respect to $|||\ \ |||$, and hence uniformly in x.

(iii) *In Cases* 1–2, *$Q(t,\cdot)$, $t>0$, has a density $q(t,x)$ with respect to the Lebesgue measure, which has the similar properties of $q_1(t,x)$ in Theorem* 1. *$u(t,x) = Q_t f(x)$ is a C^∞-function in (t,x) and satisfies* (4) *for f in B and L^1.*

4. The proofs of Theorems 1 and 2 depend on the following facts.

Lemma 1. (i) *Let μ satisfy* (2). *Then, for an arbitrary $\varepsilon > 0$, there is d_ε such that*

(12) $$|a_I(z)| \leqslant \varepsilon |z|^n + d_\varepsilon, \qquad z \in R^N.$$

(ii) *Let μ satisfy* (2) *and* (6) *for an l. Then, $D_\alpha a_I(z)$ exist for $|\alpha| \leqslant l$, and*

(13) $$|D_\alpha a_I(z)| \leqslant c_\alpha |z|^{n-(|\alpha|\wedge n)} + d_\alpha, \qquad 1 \leqslant |\alpha| \leqslant l.$$

In particular, if μ is rapidly decreasing at ∞, then $a_I(z)$ is a C^∞-function.

Lemma 2. *Let $\mathcal{R}e\,(a(z))$ be bounded from above, and let*

$$Q_t^0 f(x) = \int_{R^N} e^{ix\cdot z} e^{ta(z)} \mathcal{F}^{-1} f(dz), \qquad \text{for } f \in F_0,\ t \geqslant 0.$$

Then, $\{Q_t^0, t \geqslant 0\}$ is a semigroup of operators on each of F_l, $l = 0, 1, \cdots$, and

(14) $$\lim_{t\to 0} Q_t^0 f(x) = f(x), \qquad f \in F_0,$$

(15) $$\lim_{t\to 0} (1/t)(Q_t^0 f(x) - f(x)) = A f(x), \qquad f \in F_{m\vee n},$$

with respect to $|||\ \ |||$, and hence uniformly in x.

Lemma 3. *Let μ be given as in* (7) *and let $n \geqslant 3$. Then, μ is* *non-degenerate, if and only if*

(16) $$\mathcal{R}e\,(a_I(z)) \leqslant -c|z|^{2[(n-1)/2]}+d, \qquad z \in R^N.$$

for some positive constants c and d.

For the proof of Theorem 1, let $e_n(y,z)$ be the integrand of (5), and let

$$\mu_0(E)=\mu(E\cap\{y\,|\,0<|y|\leqslant 1\}), \qquad E\in\mathcal{B}_{R^N\setminus\{0\}}.$$

Then, $a_I(z)=a_0(z)+a_\infty(z)$, where

$$a_\infty(z)=\int_{R^N}(e^{iy\cdot z}-1)\mu_\infty(dy),$$

$$a_0(z)=\int_{R^N\setminus\{0\}} e_n(y,z)\mu_0(dy)+\int_{R^N}(1-\rho(y)\sum_{k=0}^{n-1}1/k!\,(iy\cdot z)^k)\mu_\infty(dy).$$

Under the conditions in (i) of Theorem 1, $a_0(z)$ satisfies (16) and is a C^∞-function, since μ_0 is rapidly decreasing at ∞. Hence, $e^{ta_0(z)}$ is in $\mathcal{S}$ by (12)–(13), and $e^{ta_\infty(z)}$ is the Fourier transform of (8). (ii) of Theorem 1 follows from above by a simple observation, and (iii) is a part of the well known result of Paul Lévy [2].

Theorem 2 is obtained by a simple modification of the above proof, combined with the conditions on $a_D(z)$.

The detailed proofs of the above results will be published elsewhere.

5. If $m, n\leqslant 2$, a_α's are real, (7) and (9) hold and a_0 is normalized so that $a(0)=0$, then $\{Q_t, t\geqslant 0\}$ in Theorem 2 on C_0 is the semigroup for the Lévy process in probability theory, and the converse hold.

In view of Theorems 1–2, the semigroup induced by $Q_I(t,\cdot)$ is the natural extension of the jump part of the semigroup for Lévy process for higher orders while $\{Q_t, t\geqslant 0\}$ seems to be nearly the natural extension of the semigroup for the process.

It is expected that the problem considered in this article will be solved for spatially inhomogeneous case.

We also note that the material here should be closely related with that of Gelfand-Vilenkin [1, pp. 175–188]. Their representation of a conditionally positive definite generalized function on the space Z is quite similar to the operator A here, with elliptic A_D of even order and μ given as in (7).

The result in [3] can be extended by the estimates in this article.

References

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