135. *Area Criteria for Functions to be Bloch, Normal, and Yosida*

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1. Introduction. Let $D=\{|z|<1\}$ and let

$$D(z,r)=\{w\in D\,;\,|w-z|/|1-\bar{z}w|<r\}$$

be the non-Euclidean disk of the non-Euclidean center $z\in D$ and the non-Euclidean radius $\tanh^{-1} r$, $0<r<1$. For f holomorphic in D we denote by $f(D(z,r))$ the image of $D(z,r)$ by f, namely, $f(D(z,r))$ is the set of w in the plane $C=\{|w|<\infty\}$ such that there exists $\zeta\in D(z,r)$ with $w=f(\zeta)$. Simply, $f(D(z,r))$ is the projection of the Riemannian image of $D(z,r)$ by f. Let $\alpha(z,r,f)$ be the Euclidean area of $f(D(z,r))$.

A prototype of our present study is

Theorem 1 [3]. *For f nonconstant and holomorphic in D to be Bloch, namely,*

$$\sup_{z\in D}(1-|z|^2)|f'(z)|<\infty,$$

it is necessary and sufficient that there exists r, $0<r<1$, such that

$$\sup_{z\in D}\alpha(z,r,f)<\infty.$$

We shall consider two natural analogues of Theorem 1 for normal meromorphic functions in D in the sense of O. Lehto and K. I. Virtanen [2], and for Yosida functions, namely, meromorphic functions (in the plane C) of K. Yosida's class (A) [4].

A function f meromorphic in D is said to be normal there if

$$\sup_{z\in D}(1-|z|^2)|f'(z)|/(1+|f(z)|^2)<\infty, \tag{1}$$

while a function f meromorphic in C is said to be Yosida if

$$\sup_{z\in C}|f'(z)|/(1+|f(z)|^2)<\infty. \tag{2}$$

For f meromorphic in D we let $\beta(z,r,f)$ be the spherical area of the image $f(D(z,r))$ of $D(z,r)$, $0<r<1$, contained in $C^*=C\cup\{\infty\}$, while, for f meromorphic in C we let $\gamma(z,r,f)$ be the spherical area of the image $f(\Delta(z,r))$ of the Euclidean disk $\Delta(z,r)=\{w\,;\,|w-z|<r\}$, $r>0$. Again, the images are the projections of the Riemannian images. Since C^*, regarded as the Riemann sphere of diameter one, has the spherical area π, we have two reasonable theorems, counterparts of Theorem 1.

Theorem 2. *For f nonconstant and meromorphic in D to be* *normal there, it is necessary and sufficient that there exists* r, $0<r<1$, *such that*

(3) $$\sup_{z\in D} \beta(z, r, f)<\pi.$$

Theorem 3. *For* f *nonconstant and meromorphic in* $\boldsymbol{C}$ *to be Yosida there, it is necessary and sufficient that there exists* $r>0$ *such that*

(4) $$\sup_{z\in \boldsymbol{C}} \gamma(z, r, f)<\pi.$$

2. Proofs. To prove the necessity of (3) in Theorem 2 we let $\omega(f)$ be the supremum of (1). Then, for each r, $0<r<1$,

(5) $$\beta(z, r, f)\leqq\iint_{D(z,r)} |f'(\zeta)|^2/(1+|f(\zeta)|^2)^2 d\xi d\eta$$
$$\leqq \omega(f)^2 \iint_{D(z,r)} (1-|\zeta|^2)^{-2} d\xi d\eta=\pi\omega(f)^2 r^2/(1-r^2),$$

where $\zeta=\xi+i\eta$. The second term of (5) is the area of the Riemannian image of $D(z, r)$ by f. To obtain (3) we have only to choose r with $r<(1+\omega(f)^2)^{-1/2}$.

For the proof of the sufficiency of (3) in Theorem 2 we shall make use of

Lemma. *For* g *meromorphic in the disk* $\{|z|<R\}$, $R>0$, *suppose that the spherical area* $\delta\equiv\delta(0, r, g)$ *of the image of* $\{|z|<r\}$ $(r<R)$ *by* g *is strictly less than* π. *Then,*

$$|g'(0)|^2/(1+|g(0)|^2)^2 \leqq \frac{\delta}{\pi}\Big/\left\{r^2\left(1-\frac{\delta}{\pi}\right)\right\}.$$

This is due to J. Dufresnoy [1, Lemma II, p. 216]; Dufresnoy makes use of the Riemann sphere of diameter 2, while ours is of diameter 1.

Suppose (3), and set

$$g(w)=f((w+z)/(1+\bar{z}w)).$$

A calculation then yields that

$$|g'(0)|/(1+|g(0)|^2)=(1-|z|^2)|f'(z)|/(1+|f(z)|^2).$$

Since $\delta(0, r, g)=\beta(z, r, f)$, we obtain (1), or, f is normal in D.

The proof of Theorem 3 is similar to that of Theorem 2 with minor changes.

Suppose that (2) with the supremum $\sigma(f)$ holds. Then

$$\gamma(z, r, f)\leqq\iint_{\Delta(z,r)} |f'(\zeta)|^2/(1+|f(\zeta)|^2)^2 d\xi d\eta\leqq\pi r^2\sigma(f)^2<\pi,$$

if $r<1/\sigma(f)$. Conversely, if (4) holds, then, for

$$h(w)=f(w+z)$$

we have

$$|h'(0)|/(1+|h(0)|^2)=|f'(z)|/(1+|f(z)|^2).$$

This, together with $\delta(0, r, h)=\gamma(z, r, f)$, completes the proof of the sufficiency of (4).

Remark. In the proof of Theorem 1 [3] we adopt a result of T. H. MacGregor. Since the Euclidean analogue of the above lemma of Dufresnoy is available [1, Remark, p. 216], it is now easy to prove Theorem 1 by the present method with small changes.

References

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