134. An Explicit Estimate of Sojourning Time by Intermittency with Elementary Method

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1. Introduction. First we consider iterations of a family of maps $\tilde{f}_\mu$ defined by $\tilde{f}_\mu=\mu+x-x^2$. If $\mu<0$, there are no fixed points. As $\mu\uparrow 0$, the orbit of iteration by $\tilde{f}_\mu$ more and more sojourns around 0. Finally at $\mu=0$, 0 becomes a fixed point. If $\mu>0$, there are two fixed points where the positive one is stable and the negative one is not.

Pomeau and Manneville [1], [2] studied such bifurcation in connection with an intermittent transition to turbulence in Lorenz equation. They call the motion near 0 "laminar phase" and the motion out of 0 "burst".

A general form of 1-parameter family which bifurcate as $\tilde{f}_\mu$ above is given by Guckenheimer [3]. That is,

$F(\mu, x)$ is C^3 function of both μ and x,

$$(1.1)\qquad \begin{array}{ll} (1)\ \ F(\mu_0, x_0)=x_0 & (2)\ \ \dfrac{\partial F}{\partial x}(\mu_0, x_0)=1 \\ (3)\ \ \dfrac{\partial^2 F}{\partial x^2}(\mu_0, x_0)>0 & (4)\ \ \dfrac{\partial F}{\partial \mu}(\mu_0, x_0)>0. \end{array}$$

By a conjugate transformation using an affine map and reparametrization we can transform (1.1) into the following form:

$$(1.2)\qquad F(\mu, x)=\mu+x-x^2+\mu x f(\mu, x)+x^3 g(x)$$

here $f(\mu, x)$ and $g(x)$ are continuous at the origin.

Further we introduce a noise in 1-dimensional dynamical system given by (1.2) which is different from [4] as

$$(1.3)\qquad X_{n+1}=F(\mu, X_n)+|\mu|^{1+\theta}\cdot\xi_n$$

here $\theta>0$ and ξ_n is bounded random variable.

We are interested in getting the order of increase of the duration in "laminar phase" by (1.3) as $\mu\uparrow 0$.

2. Definition and main result. Definition. Let $\delta>0$ and $F(\mu, x)$ be given by (1.2). We say δ is suitable if there exists $\mu_0<0$ such that $F(\mu, x)<x$ for $(\mu, x)\in[\mu_0, 0]\times[-\delta, \delta]-\{(0, 0)\}$.

Remark. By the definition of $F(\mu, x)$, δ which is sufficiently near 0 is suitable.

Definition. Let $\delta>0$ be suitable and consider a sequence

(2.1) $$x_0=\delta, \qquad x_{n+1}=F(\mu, x_n)+|\mu|^{1+\theta}\cdot\xi_n$$

here $\theta>0$ and there exists $C>0$ such that $|\xi_n|<C$ for all n. We define sojourning time in $[-\delta, \delta]$ denoted by $T_\delta(\mu)$ as

$$T_\delta(\mu)=\min\{n : x_n<-\delta\}.$$

Main theorem. *Let δ be suitable, then we have*

(2.2) $$\lim_{\mu\uparrow 0}\sqrt{-\mu}\cdot T_\delta(\mu)=\pi.$$

3. Proof of the main theorem. Let $\eta=\sqrt{-\mu}$ and we use $T_\delta(\eta)$ in place of $T_\delta(-\eta^2)$.

Lemma 1. *Let*

(3.1) $$\alpha>0, \qquad R(\alpha, \eta, x)=\frac{x-\alpha\eta^2}{1+\alpha x},$$

$x_0=\delta$, $x_{n+1}=R(\alpha, \eta, x_n)$ and $S_\delta(\alpha, \eta)=\min\{n : x_n<-\delta\}$. Then,

(3.2) $$S_\delta(\alpha, \eta)=\left[\frac{2\cdot\mathrm{Tan}^{-1}(\delta/\eta)}{\mathrm{Tan}^{-1}(\alpha\eta)}\right]+1.$$

Proof of Lemma 1. $R(\alpha, \eta, x)$ is projection of a rotation around $(0, -\eta)$ with angle $\mathrm{Tan}^{-1}(\alpha\eta)$ to x-axis. Consequently $R(\alpha, \eta, x)$ is conjugate to y-$\mathrm{Tan}^{-1}(\alpha\eta)$ by $y=\mathrm{Tan}^{-1}(x/\eta)$ and then $[-\delta, \delta]$ is mapped bijectively to $[-\mathrm{Tan}^{-1}(\delta/\eta), \mathrm{Tan}^{-1}(\delta/\eta)]$. Therefore we have (3.2).

Lemma 2. *Let $0<\varepsilon<1$, then there exists $\Delta>0$ such that for all $\delta\in(0, \Delta)$*

(3.3) $$\pi(1-\varepsilon)\leqq\varliminf_{\eta\downarrow 0}\eta\cdot T_\delta(\eta)\leqq\varlimsup_{\eta\downarrow 0}\eta\cdot T_\delta(\eta)\leqq\pi(1+\varepsilon).$$

Proof of Lemma 2. Let

$$h(x, \eta)=\frac{\eta^2 x\cdot f(-\eta^2, x)-x^3 g(x)}{\eta^2+x^2},$$

then $h(0, 0)=0$ and $h(x, \eta)$ is continuous at $(0, 0)$. From (1.2) and (3.1),

$$R\left(\frac{1}{1+\varepsilon}, \eta, x\right)-F(-\eta^2, x)=(x^2+\eta^2)\left(\frac{\varepsilon+x}{1+\varepsilon+x}+h(x, \eta)\right)$$

$$R\left(\frac{1}{1-\varepsilon}, \eta, x\right)-F(-\eta^2, x)=(x^2+\eta^2)\left(\frac{-\varepsilon+x}{1-\varepsilon+x}+h(x, \eta)\right).$$

As $\varepsilon/(1+\varepsilon)>0$, $-\varepsilon/(1-\varepsilon)<0$ and $x^2+\eta^2\geqq 0$, there exist Δ, $\eta_*>0$ such that

(3.4) $$R\left(\frac{1}{1-\varepsilon}, \eta, x\right)\leqq F(-\eta^2, x)\leqq R\left(\frac{1}{1+\varepsilon}, \eta, x\right)$$

for any $(\eta, x)\in[-\eta_*, \eta_*]\times[-\Delta, \Delta]$. Now $|\xi_n|$ is bounded, hence for sufficiently large $A>0$ we have

(3.5) $$-\frac{A|\eta|^{2+2\theta}}{1-\varepsilon+x}<\eta^{2+2\theta}\cdot\xi_n<\frac{A|\eta|^{2+2\theta}}{1+\varepsilon+x}$$

for $x\in[-\Delta, \Delta]$. Then by (3.4) and (3.5) we have

(3.6) $$R\left(\frac{1}{1-\varepsilon}, \eta_2, x\right)<\eta^{2+2\theta}\cdot\xi_n+F(-\eta^2, x)<R\left(\frac{1}{1+\varepsilon}, \eta_1, x\right)$$

here $\eta_1=\eta\sqrt{1+A\eta^{2\theta}}$ and $\eta_2=\eta\sqrt{1-A\eta^{2\theta}}$. In (3.6) the left-hand side decreases faster than F in $[-\Delta, \Delta]$ and the right-hand side decreases slower than F in $[-\Delta, \Delta]$. Therefore for any $\delta \in (0, \Delta)$ we have

$$S_\delta\left(\frac{1}{1-\varepsilon}, \eta_2\right) \leqq T_\delta(\eta) \leqq S_\delta\left(\frac{1}{1+\varepsilon}, \eta_1\right).$$

Both sides are given by (3.2) and we have (3.3) through elementary calculation.

Proof of the main theorem. Let δ be suitable and $0<\delta'<\delta$, then by definition

$$M=\min\{|x-F(\mu, x)-|\mu|^{1+\theta}\cdot C| : (\mu, x) \in [\mu_*, 0]\times[\delta', \delta]\cup[-\delta, -\delta']\}$$

exists and is positive for some μ_*. Hence the sojourning time in $[\delta', \delta]$ and in $[-\delta, -\delta']$ are bounded by $(\delta-\delta')/M$. Therefore we have

$$T_\delta(\mu)-2(\delta-\delta')/M \leqq T_{\delta'}(\mu) \leqq T_\delta(\mu).$$

Hence we have

$$\varliminf_{\mu\uparrow 0} \sqrt{-\mu}\cdot T_{\delta'}(\mu) \leqq \varliminf_{\mu\uparrow 0} \sqrt{-\mu}\cdot T_\delta(\mu) \leqq \varlimsup_{\mu\uparrow 0} \sqrt{-\mu}\cdot T_\delta(\mu) \leqq \varlimsup_{\mu\uparrow 0} \sqrt{-\mu}\cdot T_{\delta'}(\mu).$$

By Lemma 2, both sides are bounded by $\pi(1-\varepsilon)$ and $\pi(1+\varepsilon)$ respectively. Hence we have (2.2).

Remark. The key idea of the proof of our theorem is that $F(\mu, x)$ satisfying (1.2) is well approximated by $R(1, \sqrt{-\mu}, x)=(x+\mu)/(1+x)$ which is essentially a rotation around $(0, -\sqrt{-\mu})$.

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