14. *Surjectivity for a Class of Dissipative Operators*

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Let X be a real Banach space with the norm $\|\cdot\|$. We will call a mapping $A: X\to 2^X$ a (multi-valued) operator in X. The domain of A is the set $D(A)=\{x\in X;\ Ax\neq\phi\}$ and the range of A is the set $R(A)=\bigcup_{x\in D(A)} Ax$. We also define the generalized domain $D_a(A)$ of A as the set of $x\in X$, for which there exist sequences x_n, y_n and a positive number M such that $x_n\to x$ as $n\to\infty$, $y_n\in Ax_n$ and $\|y_n\|\leqq M$ for $n=1,2,\cdots$. We denote the infimum of such numbers M by $|Ax|$ for each $x\in D_a(A)$. It is easy to show that $D(A)\subset D_a(A)\subset\overline{D(A)}$ and the set $\{x\in D_a(A);\ |Ax|\leqq r\}$ is closed in X for each $r\geqq 0$.

For operators A and B in X and scalars λ and μ, we define the operator $\lambda A+\mu B$ by setting $(\lambda A+\mu B)x=\{\lambda y+\mu z;\ y\in Ax \text{ and } z\in Ax\}$ for $x\in D(A)\cap D(B)$ and $D(\lambda A+\mu B)=D(A)\cap D(B)$. An operator A in X is single-valued if Ax is singleton for $x\in D(A)$ and, in this case, we use Ax to denote both the singleton and its element. Let I denote the identity operator in X.

An operator A in X is said to be dissipative if

$$\|x_1-x_2\|\leqq\|x_1-x_2-\lambda(y_1-y_2)\|$$

for $\lambda>0$, $x_i\in D(A)$ and $y_i\in Ax_i$, $i=1,2$. A dissipative operator A in X is said to be m-dissipative if $R(I-\lambda A)=X$ for $\lambda>0$.

The following theorem is proved in [4], Corollary 2.

Theorem 1. *Let A be a dissipative operator in X. Then the following* (i) *and* (ii) *are equivalent to each other.*

(i) *A is m-dissipative.*

(ii) *For each $x\in D_a(A)$ and $w\in X$, there exists a sequence $\delta_n>0$ such that $\delta_n\to 0$ as $n\to\infty$ and $R(I-\delta_n(A+w))\ni x$ for $n=1,2,\cdots$.*

As an application of this result, the following theorem due to Browder [1] can be readily proved.

Theorem 2. *Let A be a single-valued dissipative operator in X which is locally Lipshitz continuous on $X=D(A)$. Then A is m-dissipative.*

Some other applications of Theorem 1 can be found in [4]. Recently, in [5], Ray gave a simple proof of above Theorem 2, by using Caristi's fixed point theorem established in [2]. The purpose of this paper is to show that the Ray's argument also applies to the proof of Theorem 1. In fact, we establish the following general lemma, from which Theorem 1 is derived through a standard argument.

Lemma 1. *Let B be an operator in X such that $B+I$ is dissipative. Assume that for each $x \in D_a(B)$, there exist $\mu>0$ and $x_\mu \in D(B)$ such that $x_\mu - \mu B x_\mu \ni x$. Then there exists an element $X_0 \in D(B)$ such that $Bx_0 \ni 0$.*

Proof. First set a positive number r so that $X_0=\{x \in D_a(B)\,;\,|Bx| \leqq r\}$ is not empty and take $\phi(x)=|Bx|$ for $x \in X_0$. Then, $\phi(x)$ is lower semi-continuous on X_0 and X_0 is closed in X. Let $x \in X_0$. Then, by our assumption, there exist $\mu>0$ and $x_\mu \in D(B)$ such that $x_\mu - \mu B x_\mu \ni x$. We want to show that

$$x_\mu \in X_0 \quad \text{and} \quad \|x_\mu - x\| \leqq \phi(x)-\phi(x_\mu). \tag{1}$$

Once this is proved, then it follows from Caristi's fixed point theorem in [2] that there exist $\mu>0$ and $x_0 \in X_0 \cap D(B)$ such that $x_0-\mu B x_0 \ni x_0$, i.e., $Bx_0 \ni 0$. To show (1), let $y_\mu = \mu^{-1}(x_\mu - x) \in Bx_\mu$ and take $u_n \in D(B)$ and $v_n \in Bu_n$ so that $u_n \to x$ and $\|v_n\| \to |Bx|$ as $n \to \infty$. Since $B+I$ is dissipative, it follows that

$$\|x_\mu - u_n\| \leqq \|x_\mu - u_n - \mu(1+\mu)^{-1}(y_\mu + x_\mu - (v_n+u_n))\|$$

or

$$(1+\mu)\|x_\mu - u_n\| \leqq \|x_\mu - u_n - \mu(y_\mu - v_n)\|.$$

Letting $n \to \infty$, we have

$$(1+\mu)\|x_\mu - x\| \leqq \|x_\mu - x - \mu y_\mu\| + \mu|Bx| = \mu|Bx|. \tag{2}$$

This implies

$$|Bx_\mu| \leqq \|y_\mu\| = \mu^{-1}\|x_\mu - x\| \leqq (1+\mu)^{-1}|Bx| \leqq (1+\mu)^{-1} r \leqq r, \tag{3}$$

that is, $x_\mu \in X_0$. Furthermore, (2) and (3) together imply

$$\mu\|x_\mu - x\| \leqq \mu|Bx| - \|x_\mu - x\| \leqq \mu|Bx| - \mu|Bx_\mu|.$$

Thus we get (1). Q.E.D.

Proof of Theorem 1. Evidently, (i) implies (ii). Assume (ii) holds and let $y \in X$ and $\lambda>0$. Define an operator B in X by setting $B=\lambda A - I + y$. Then $B+I$ is dissipative and $D_a(B)=D_a(A)$. Let $x \in D_a(B)$. Then the assumption states that there exist $\delta \in (0,\lambda)$ and $x_\delta \in D(A)$ such that

$$x_\delta - \delta(Ax_\delta + \lambda^{-1}(y-x)) \ni x.$$

Choose $\mu>0$ so that $\delta = \lambda\mu/(1+\mu)$. Then

$$x_\delta - \mu(\lambda A x_\delta - x_\delta + y) \ni x$$

or

$$x_\delta - \mu B x_\delta \ni x.$$

Thus Lemma 1 implies that there exists an element $x_0 \in D(B)=D(A)$ such that $Bx_0 \ni 0$, i.e., $x_0 - \lambda A x_0 \ni y$. Q.E.D.

References

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