127. On Powers of the Denominators of Rationals

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For a rational x let den x mean its reduced denominator >0. We denote by $\mathrm{den}^\alpha x$ the αth power of den x, where α is a complex number. In this note we show some properties of the function $\mathrm{den}^\alpha x$.

Theorem 1. *Let a, b, c, d be rational integers with $ad-bc=1$. Then for any rational x with $cx+d\neq 0$, we have*

$$(1)\qquad \mathrm{den}^\alpha \frac{ax+b}{cx+d}=|cx+d|^\alpha \,\mathrm{den}^\alpha x.$$

Conversely, if a function $f(x)$ defined on the rationals satisfies the functional equations

$$(*)\qquad f(x+1)=f(x)$$

$$(**)\qquad f\left(-\frac{1}{x}\right)=|x|^\alpha f(x) \qquad (x\neq 0),$$

then

$$f(x)=f(0)\,\mathrm{den}^\alpha x.$$

Theorem 2. *For every positive integer n, we have*

$$(2)\qquad n^{\alpha-1}\sum_{\substack{ad=n\\ d>0}}\sum_{b=0}^{d-1} d^{-\alpha}\,\mathrm{den}^\alpha \frac{ax+b}{d}=\frac{1}{n}\sigma_{\alpha+1}(n)\,\mathrm{den}^\alpha x,$$

where $\sigma_{\alpha+1}(n)$ is the sum of the $(\alpha+1)$th powers of positive divisors of n.

Theorem 3. *For every positive integer n, we have*

$$(3)\qquad \sum_{b=0}^{n-1}\mathrm{den}^\alpha \frac{x+b}{n}=\sum_{d\mid n}\mu(d)d^\alpha\sigma_{\alpha+1}\left(\frac{n}{d}\right)\mathrm{den}^\alpha dx,$$

where μ means Möbius' function.

Proof of Theorem 1. The transformation property (1) is easily verified. We prove the second assertion by induction on den x. Write x in the form h/k where h and k are relatively prime integers and $k>0$. When den $x=k=1$, namely x is a rational integer, we see immediately by the equation $(*)$

$$f(x)=f(0)=f(0)\,\mathrm{den}^\alpha x.$$

When den $x=k>1$, we assume that our assertion is valid for any rational with the reduced denominator less than k. We may consider $0<x<1$ because of $(*)$. Then, using the equation $(**)$, we find

$$f(x)=f\left(\frac{h}{k}\right)=f\left(-\frac{1}{-k/h}\right)=\left|-\frac{k}{h}\right|^\alpha f\left(-\frac{k}{h}\right)$$
$$=\left(\frac{k}{h}\right)^\alpha h^\alpha f(0)=k^\alpha f(0)=f(0)\,\mathrm{den}^\alpha x$$

by the induction hypothesis, since $0<h<k$. This completes the proof.

Proofs of Theorems 2 *and* 3. From the multiplicative property of $\sigma_{\alpha+1}(n)$ and the usual properties of the Hecke operator and the averaging operator which are defined on $\mathrm{den}^{\alpha} x$ by the left sides of (2) and (3) respectively [1], [5], [7], Theorems 2 and 3 reduce to the following lemma.

Lemma. *For a prime number p*

$$p^{\alpha}\,\mathrm{den}^{\alpha}\,px+\sum_{b=0}^{p-1}\mathrm{den}^{\alpha}\,\frac{x+b}{p}=\sigma_{\alpha+1}(p)\,\mathrm{den}^{\alpha}\,x. \tag{4}$$

Proof. We write $x=h/k$ as above. To verify (4) we must consider two cases:

Case I. p divides k. Then $\mathrm{den}\,px=k/p$. On the other hand $(x+b)/p=(h+bk)/pk$, and $h+bk\equiv h\not\equiv 0\ (\mathrm{mod}\ p)$, so that $\mathrm{den}\,((x+b)/p)=pk$. Hence the left-hand side of (4) equals

$$p^{\alpha}\Big(\frac{k}{p}\Big)^{\alpha}+p(pk)^{\alpha}=(1+p^{\alpha+1})k^{\alpha}=\sigma_{\alpha+1}(p)\,\mathrm{den}^{\alpha}\,x.$$

Case II. p does not divide k. Then $\mathrm{den}\,px=k$. Since $h+bk\equiv 0\ (\mathrm{mod}\ p)$ if and only if $b\equiv -hk^{-1}\ (\mathrm{mod}\ p)$, we see

$$\mathrm{den}\,\frac{x+b}{p}=\begin{cases}k & \text{if } b\equiv -hk^{-1}\ (\mathrm{mod}\ p),\\ pk & \text{otherwise.}\end{cases}$$

Hence the left-hand side of (4) equals

$$p^{\alpha}k^{\alpha}+k^{\alpha}+(p-1)(pk)^{\alpha}=\sigma_{\alpha+1}(p)\,\mathrm{den}^{\alpha}\,x.$$

Remarks. Theorems 2 and 3 are equivalent by the Möbius inversion formula [7]. The above (1)–(4) respectively correspond to the generalized reciprocity formula (the transformation formula), the identities of Petersson-Knopp type, of Subrahmanyam type, and of Dedekind type for a kind of generalized Dedekind sums (cf. the references). In fact, $B_{2k}\,\mathrm{den}^{-2k}\,x$ with a positive integer k and the $2k$th Bernoulli number B_{2k} is an inhomogeneous generalized Dedekind sum [7].

References

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