126. *On Quartic Surfaces and Sextic Curves with Certain Singularities**)

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This is a summary of our recent results on singularities on algebraic varieties. Details will appear elsewhere. We show that for quartic surfaces and sextic curves with singularities $\tilde{E}_7$, $T_{2,4,5}$, Z_{11}, $\tilde{E}_6$, $T_{3,3,4}$ and Q_{10}, the configurations of the singularities are expressible in terms of Dynkin graphs (cf. Arnold [1], Saito [4], Bourbaki [2]). In this article we assume that every variety is algebraic and defined over the complex number field C.

Definition. For a given set of Dynkin graphs, we call the following procedure *an elementary transformation.*

(1) Replace each component by the extended Dynkin graph of the corresponding type.

(2) Choose in an arbitrary manner at least one vertex from each component (of the extended Dynkin graph) and then remove these vertices together with the edges issuing from them.

Note that any Dynkin graph without multiple edges is associated with a rational double point on a surface (cf. Durfee [3]).

Theorem 1. *Consider a normal quartic surface X in the three dimensional projective space P^3.*

(I) *Assume that X has a simple elliptic singularity $\tilde{E}_7$. Then the configuration of the singularities on X is $\tilde{E}_7$ plus one of the following:*

(1) *a configuration of rational double points associated with a set of Dynkin graphs obtained from E_7+B_3 by elementary transformations repeated twice such that the resulting set has no vertices corresponding to short roots,*

(2) *another $\tilde{E}_7$ plus a configuration of rational double points associated with a subgraph of the Dynkin graph A_3.*

Conversely, any one of the above configurations plus $\tilde{E}_7$ can be realized on a normal quartic surface as the configuration of its singularities.

(II) *Let X have a cusp singularity $T_{2,4,5}$. Then the configuration of the singularities on X is $T_{2,4,5}$ plus a configuration of rational double points associated with a set of Dynkin graphs obtained from E_7+B_3 by one elementary transformation such that the resulting set has no*

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vertex corresponding to a short root. Conversely, any one of such configurations of rational double points plus $T_{2,4,5}$ *can be realized on a normal quartic surface.*

(III) *If X has a unimodular exceptional singularity* Z_{11}, *then the configuration of singularities on X is* Z_{11} *plus a configuration of rational double points associated with a subgraph of the Dynkin graph* E_7+A_2. *Conversely, any one of such configurations of rational double points plus* Z_{11} *can be realized on a normal quartic surface.*

(IV) *Suppose that X has a simple elliptic singularity* $\tilde{E}_6$. *Then the configuration of the singularities on X is* $\tilde{E}_6$ *plus one of rational double points associated with a subgraph of the Dynkin graph* A_{11}. *Conversely,* $\tilde{E}_6$ *plus rational double points associated with a subgraph of* A_{11} *can be realized on a normal quartic surface.*

(V) *Let X have a cusp singularity* $T_{3,3,4}$. *Then the singularities on X are* $T_{3,3,4}$ *plus some rational double points associated with a subgraph of* A_{11}. *Conversely,* $T_{3,3,4}$ *plus any number of rational double points associated with a subgraph of* A_{11} *can be realized on a normal quartic surface.*

(VI) *Similarly if X has a unimodular exceptional singularity* Q_{10}, *then the singularities on X are* Q_{10} *plus those associated with a subgraph of* A_{11}. *Conversely,* Q_{10} *plus a configuration associated with a subgraph of* A_{11} *can be realized on a normal quartic surface.*

Remark 1. Consider a smooth irreducible elliptic curve D on a smooth surface S whose self-intersection number D^2 is -2 or -3. The normal isolated singularity obtained by contracting D to a point is then $\tilde{E}_7$ or $\tilde{E}_6$ according as D^2 is -2 or -3. Their local defining equations have the following normal forms (cf. Saito [4]).

$$\tilde{E}_7: z^2+xy(x-y)(x-ay)=0, \qquad a\neq 0,1,$$
$$\tilde{E}_6: z^2y+x(x-y)(x-ay)=0, \qquad a\neq 0,1.$$

Remark 2. Let D be an irreducible rational curve on S whose singularity is an ordinary double point. We assume that $D^2=-2$ (or $D^2=-3$). The normal isolated singularity obtained by contracting D to a point is $T_{2,4,5}$ (or $T_{3,3,4}$) with the following normal form of the local defining equation (cf. Arnold [1]):

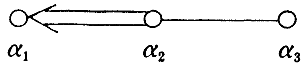
$$T_{2,4,5}: x^2+y^4+z^5-xyz=0,$$
$$T_{3,3,4}: x^3+y^3+z^4-xyz=0.$$

Remark 3. Given an irreducible rational curve D on S whose singularity is an ordinary cusp. Assume $D^2=-2$ (or $D^2=-3$). The normal isolated singularity obtained by contracting D to a point is Z_{11} (or Q_{10}) with the following normal form of the defining equation:

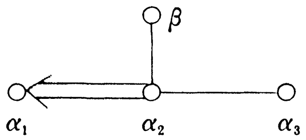
$$Z_{11}: x^3y+y^5+z^2+axy^4=0,$$
$$Q_{10}: x^3+y^4+yz^2+axy^3=0, \qquad a\in \boldsymbol{C}.$$

Remark 4. Part (I), (2) has already been obtained by Y. Umezu in [5].

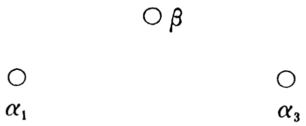
Remark 5. Consider the Dynkin graph B_3.



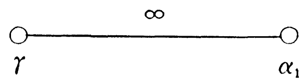
The vertex α_1 corresponds to a short root. Now consider the following case in particular. We erase α_2, but keep α_1 in the extended Dynkin graph of B_3,



obtaining the graph



(here α_3 and β may or may not have been erased). This is an elementary transformation. Next, we apply another elementary transformation. In the extended Dynkin graph, the new vertex γ joined to α_1 must be regarded as a short root.



Thus both γ and α_1 have to be erased in the second step of this elementary transformation.

Now recall that if two power series $z^2+f(x,y)$, $z^2+g(x,y)$ with $f, g \in \boldsymbol{C}\{x, y\}$ can be transformed in $\boldsymbol{C}^3$ to each other by an analytic coordinate change around the origin, then f and g themselves can also be transformed in $\boldsymbol{C}^2$ to each other by an analytic coordinate change around the origin. Thus we shall call the singularity defined by $f(x,y)=0$ by the same name as the one defined by $z^2+f(x,y)=0$. Under this convention, we can use such phrase as "a plane curve singularity of type $\tilde{E}_7$" etc.

Theorem 2. *Consider a reduced sextic curve B in the two dimensional projective space* $\boldsymbol{P}^2$.

(I) *Assume that B has a simple elliptic singularity* $\tilde{E}_7$. *Then the configuration of the singularities on B is* $\tilde{E}_7$ *plus one of rational double points associated with a set of Dynkin graphs obtained from the Dynkin graph* D_{10} *by elementary transformations repeated twice. Conversely, any one of such configuration of rational double points plus* $\tilde{E}_7$ *can be realized on a reduced sextic curve.*

(II) *Let B have a cusp singularity* $T_{2,4,5}$. *Then the configuration of singularities on B is* $T_{2,4,5}$ *plus one of rational double points associated with a set of Dynkin graphs obtained from* D_{10} *by one elementary transformation. Conversely, any one of such configuration of rational double points plus* $T_{2,4,5}$ *can be realized on a reduced sextic curve.*

(III) *If B has a unimodular exceptional singularity* Z_{11}, *then the singularities on B is* Z_{11} *plus some rational double points associated with a subgraph of the Dynkin graph* D_{10}. *Conversely, any subgraph of* D_{10} *plus* Z_{11} *can be realized on a reduced sextic curve as a configuration of singularities.*

These results are a continuation of our previous ones in [6], in which we treated the cases of quartic surfaces and sextic curves with singularities $\tilde{E}_8$, $T_{2,3,7}$ and E_{12}.

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