# 121. On the Structure of Solutions to the Self-Dual Yang-Mills Equations

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The purpose of this note is to establish a new formulation of the "complete integrability" of the self-dual Yang-Mills (SDYM) equations to the effect that by introducing infinitely many new dependent variables the SDYM equations are transformed into equations which can be solved easily.

**1. Introducing new dependent variables.** In the four dimensional complex flat space $C^4$ with coordinates $x=(y, z, \bar{y}, \bar{z})$, the SDYM equations with structure group $GL(r, C)$ $(r\geq 2)$ read

$$(1)\qquad [\partial_y+A_y,\ \partial_z+A_z]=0, \qquad [\partial_{\bar{y}}+A_{\bar{y}},\ \partial_{\bar{z}}+A_{\bar{z}}]=0,$$
$$[\partial_y+A_y,\ \partial_{\bar{y}}+A_{\bar{y}}]+[\partial_z+A_z,\ \partial_{\bar{z}}+A_{\bar{z}}]=0,$$

where $\partial_u=\partial/\partial u$, $u=y, z, \bar{y}, \bar{z}$, and $A_u$, $u=y, z, \bar{y}, \bar{z}$, denote the $gl(r, C)$-valued unknown functions depending on $x$. Contrary to the usual formulation, no reality conditions are imposed. Then $A_y$ and $A_z$ are eliminated by a suitable complex gauge transformation $A_u\to G^{-1}A_uG+G^{-1}\partial_uG$, $u=y, z, \bar{y}, \bar{z}$, $G=G(x)$. Under this gauge-fixing condition (1) reduces to the equations

$$(2)\qquad \partial_{\bar{y}}A_{\bar{z}}-\partial_{\bar{z}}A_{\bar{y}}+[A_{\bar{y}}, A_{\bar{z}}]=0, \qquad \partial_yA_{\bar{y}}+\partial_zA_{\bar{z}}=0.$$

In what follows all the formal power series solutions $A_{\bar{y}}, A_{\bar{z}} \in gl(r, C[[x]])$ to (2) are considered. (It is also possible to generate all the local holomorphic solutions defined at $x=0$ by adding some analytical conditions to the following argument, though we shall not discuss them here.)

To seek for an explicit description of all the formal power series solutions to (2), we introduce infinitely many new dependent variables to transform (2) consistently, namely, without adding any essentially new conditions which may exclude some of the solutions, into new equations. This procedure is carried out in two steps.

The first step is due to a modification of the observation of Belavin and Zakharov [1]. It is pointed out in [1] that (2) are nothing but the integrability (compatibility) conditions of the linear system

$$(3)\qquad (-\lambda\partial_y+\partial_{\bar{z}}+A_{\bar{z}})W=0, \qquad (\lambda\partial_z+\partial_{\bar{y}}+A_{\bar{y}})W=0.$$

In our case we require $W=W(x, \lambda)$ to be a formal power series of the form $W=\sum_{j=0}^{\infty} W_j\lambda^{-j}$ with $W_j \in gl(r, C[[x]])$ and $W_0=1_r$, the $r\times r$ unit matrix. In terms of $W_j$, equations (3) read

(4) $-\partial_y W_{j+1}+\partial_{\bar z} W_j+A_{\bar z} W_j=0, \quad \partial_z W_{j+1}+\partial_{\bar y} W_j+A_{\bar y} W_j=0, \quad j\geq 0.$

To obtain $W_j$, we must solve (4) recursively ; (2) implies the integrability conditions to solve (4). Hence,

**Proposition 1.** *For any solution $A_{\bar y}, A_{\bar z}\in gl(r, \boldsymbol{C}[[x]])$ to* (2) *there exists a solution to* (4) *with $W_j\in gl(r, \boldsymbol{C}[[x]])$, $W_0=1_r$. Conversely,* (2) *follows from* (4).

The equations for $j=0$ in (4) imply that $A_{\bar y}$ and $A_{\bar z}$ are regained by the formulas

(5) $$A_{\bar y}=-\partial_z W_1, \qquad A_{\bar z}=\partial_y W_1.$$

Finally, substituting (5) to (4) we obtain the differential equations

(6) $$\begin{aligned}&-\partial_y W_{j+1}+\partial_{\bar z} W_j+(\partial_y W_1)W_j=0,\\ &\partial_z W_{j+1}+\partial_{\bar y} W_j-(\partial_z W_1)W_j=0, \qquad j\geq 0,\end{aligned}$$

for the new dependent variables $W_j$. Proposition 1 shows that all the formal power series solutions to (2) are derived from those to (6) via (5).

The second step is achieved by introducing a one-to-one correspondence between $W$ and an $\infty\times\infty$ matrix

$$\xi=(\xi_{ij})_{i\in \boldsymbol{Z}, j<0}=\begin{pmatrix} & \cdots\cdots \\ \cdots & \xi_{-1,-2}\ \xi_{-1,-1} \\ \cdots & \xi_{0,-2}\ \ \xi_{0,-1} \\ & \cdots\cdots \end{pmatrix}, \qquad \xi_{ij}\in gl(r, \boldsymbol{C}[[x]]),$$

which satisfies the conditions

(7) $$\xi_{ij}=\delta_{ij}1_r \qquad \text{for } i<0,\ j<0,$$

(8) $$\Lambda\xi=(\xi_{i+1,j})_{i\in \boldsymbol{Z}, j<0}=\xi C, \qquad C=\begin{pmatrix}(\delta_{i+1,j})_{i<-1,j<0}\\ (\xi_{0j})_{j<0}\end{pmatrix},$$

where $\Lambda=(\delta_{i+1,j}1_r)_{i,j\in \boldsymbol{Z}}$, and $\delta_{ij}$ denotes Kronecker's delta. The correspondence is defined by

(9) $$\xi_{0j}=-W_{-j}, \qquad j<0.$$

Since by virtue of (7) and (8) $\xi$ is uniquely determined by $\xi_{0j}$, $j<0$, (9) actually defines a correspondence $W\leftrightarrow\xi$. Further,

**Proposition 2.** *We have*

(10) $$\xi=(W^*_{i-j})_{i\in \boldsymbol{Z}, j<0}(W_{i-j})_{i,j<0},$$

*where $W_j=W^*_j=0$ for $j<0$, and $W^*_j$, $j\geq 0$, denote the coefficients of $W^{-1}$,* i.e. $W^{-1}=\sum_{j=0}^{\infty} W^*_j\lambda^{-j}$. (Note that $W^*_j$ are recursively calculated by the formula $W^*_j=-\sum_{k=0}^{j-1} W^*_k W_{j-k}+\delta_{j0}1_r$.)

It follows immediately from (10) that $\xi_{ij}=G_{i+1,-j}$ for $i\geq 0$ and $j<0$, where $G_{ij}$, $i\geq 1$, $j\geq 1$, denote the characteristic matrices of $W$ introduced by Jimbo and Miwa [2]; recall that they are originally defined by the formula $W(x,\mu)^{-1}W(x,\lambda)-1_r=(\lambda-\mu)\sum_{i,j=1}^{\infty} G_{ij}\mu^{-i}\lambda^{-j}$ with indeterminate variables $\lambda$ and $\mu$. Now, we have

**Proposition 3.** *Via the correspondence $W\leftrightarrow\xi$, equations* (6) *are equivalent to*

$$(11)\qquad (-\Lambda\partial_y+\partial_{\bar z})\xi+\xi A=0, \qquad (\Lambda\partial_z+\partial_{\bar y})\xi+\xi B=0,$$
$$A=\begin{pmatrix} 0 \\ (\partial_y\xi_{0j})_{j<0}\end{pmatrix}, \qquad B=\begin{pmatrix} 0 \\ (-\partial_z\xi_{0j})_{j<0}\end{pmatrix}.$$

Thus the problem of describing all the formal power series solutions to (2) is converted into solving equations (11) for $\xi=(\xi_{ij})_{i\in Z,j<0}$, $\xi_{ij}\in gl(r, C[[x]])$, under conditions (7) and (8).

The aspect of the initial value problem with respect to the plane $\bar y=\bar z=0$ provides a convenient framework for the description of the solutions. In fact, it is easy to see that any solution $\xi$ to (11) satisfying (7) and (8) is uniquely determined by its initial value $\xi^{(0)}=(\xi^{(0)}_{ij})_{i\in Z,j<0}=\xi|_{\bar y=\bar z=0}$, $\xi^{(0)}_{ij}\in gl(r, C[[y,z]])$; $\xi^{(0)}$ is required only to fulfill conditions (7) and (8) in place of $\xi$. In other words, the solution space is faithfully parametrized by the space of initial values.

**2. Solving the initial value problem.** The initial value problem presented above is solved easily. Let $\xi^{(0)}=(\xi^{(0)}_{ij})_{i\in Z,j<0}$, $\xi^{(0)}_{ij}\in gl(r, C[[y,z]])$, be an arbitrary initial value satisfying (7) and (8). We set

$$(12)\qquad \tilde\xi=(\tilde\xi_{ij})_{i\in Z,j<0}=\exp(\bar z\Lambda\partial_y-\bar y\Lambda\partial_z)\xi^{(0)}, \qquad \tilde\xi_{(-)}=(\tilde\xi_{ij})_{i,j<0}.$$

Then it follows that the inverse $\tilde\xi^{-1}_{(-)}$ can be constructed, for example, by using some Neumann series, to be again an $\infty\times\infty$ matrix consisting of $r\times r$ blocks $\in gl(r, C[[x]])$, and that the product $\tilde\xi\tilde\xi^{-1}_{(-)}$ makes sense similarly. Finally,

**Theorem 4.** *The matrix*

$$(13)\qquad \xi=\tilde\xi\tilde\xi^{-1}_{(-)}$$

*is the solution to the initial value problem with initial value* $\xi^{(0)}$. *Namely,* $\xi$ *satisfies* (7), (8), (11) *and the initial condition* $\xi|_{\bar y=\bar z=0}=\xi^{(0)}$.

The above result reveals the very simple structure of the evolution $\xi^{(0)}\to\xi$. Note that the essential part of the evolution is given by the simple operator $\exp(\bar z\Lambda\partial_y-\bar y\Lambda\partial_z)$. Our construction of solutions can be regarded as a generalization of Sato's method [4], where the counterpart of the matrix $\xi$ appears as a frame matrix representing a moving point of the infinite dimensional Grassmann manifold. A close relationship with Mulase's method [3] can be also pointed out. In fact, it can be shown that $W$ itself, corresponding to $\xi$, is uniquely characterized in terms of its initial value $W^{(0)}=W|_{y=z=0}$ by the condition

$$(14)\qquad W\exp(\bar z\lambda\partial_y-\bar y\lambda\partial_z)W^{(0)-1}\in gl(r, C[[x]][[\lambda]]);$$

this parallels Mulase's construction of solutions to the KP hierarchy.

In conclusion, we may say that the SDYM equations are "completely integrable" in the spirit of Sato [4].

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## References

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