13. Non-Uniqueness in the Cauchy Problem for Partial Differential Operators with Multiple Characteristics

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The purpose of this paper is to give a necessary condition for uniqueness of C^∞-solutions of the non-characteristic Cauchy problem for a class of partial differential operators with multiple characteristics of constant multiplicity or of variable multiplicity. For operators with multiple characteristics, many authors obtained various sufficient conditions for uniqueness on the lower order terms (see [1], [4], [5], [7]). However, except [1], it is still unclear whether their conditions are necessary or not. We shall give an answer to this question. The author believes that our results will clarify a role of lower order terms in this theory.

§ 1. Statement of results. We consider the following operator in $\boldsymbol{R}^{d+1}$:

$$P=P(t, x\,;\partial_t, D_x)=\partial_t^p+t^k A(t, x\,; D_x)-t^m B(t, x\,; D_x),$$

where $\partial_t=\partial/\partial t$, $D_x=(\partial/i\partial x_1, \cdots, \partial/i\partial x_d)$, $k, m \in \boldsymbol{N}$, A and B are partial differential operators with respect to x of order q and $q-r$ respectively ($p\geqq q>r\geqq 1$) with C^∞-coefficients in U, an open neighborhood of the origin in $\boldsymbol{R}^{d+1}$. Let $A_q(t, x\,;\xi)$ and $B_{q-r}(t, x\,;\xi)$ be the principal symbols of A and B respectively. We assume

$$\text{(A.1)} \qquad k>(pr+qm)/(q-r).$$

We also assume that there exist $\xi^0 \in \boldsymbol{R}^d\backslash\{0\}$ and a root $C=C(\xi^0)$ of the equation $X^p=B_{q-r}(0,0\,;\xi^0)-A_q(0,0\,;\xi^0)$ satisfying

$$\text{(A.2)} \qquad \operatorname{Re} C(\xi^0)>0,$$

$$\text{(A.3)} \qquad \operatorname{Re}\left\{\left(\frac{A_q(0,0\,;\xi^0)}{B_{q-r}(0,0\,;\xi^0)-A_q(0,0\,;\xi^0)}+1-\frac{q}{r}\right)C(\xi^0)\right\}>0.$$

Now, we state the main theorem.

Theorem 1. *Under assumptions* (A.1)–(A.3), *there exist an open neighborhood U' of the origin and $u, f \in C^\infty(U')$ satisfying*

$$Pu-fu=0, \qquad (0,0)\in \operatorname{supp} u\subset\{t\geqq 0\}.$$

Next, we consider the following operator:

$$P'=P+\sum_{i+j\leqq p}' t^{k(i,j)}B_{i,j}(t, x\,; D_x)\partial_t^j,$$

where P is the operator treated above and $B_{i,j}$ are operators of order i with coefficients in $C^\infty(U)$. We assume

(A.4) $$k(i,j)>k(1-j/p)+(k-m)(i-q+jq/p)/r.$$

Then we have

Theorem 2. *Under assumptions* (A.1)–(A.4), *the same conclusion as in Theorem* 1 *holds for* P'.

Remark 1. When $p=q$ and $k=pl$, (A.1) implies $m<l(p-r)-r$. On the other hand, Uryu [7] showed that uniqueness holds for such P if $m\geqq l(p-r)-r$. Hence assumption (A.1) is the best one.

Remark 2. Even when $p>q$, assumption (A.1) seems to be the best. In fact, we have

Theorem 3. *Let Q be the operator in $\boldsymbol{R}^2$*:

$$Q=\partial_t^p+t^kA(t,x)D_x^2-t^mB(t,x)D_x+C(t,x),$$

where $p=3$ or 4, k, $m\in N$, $A, B, C\in C^\infty(U)$. We assume

(A.5) $$k\leqq p+2m$$

(A.6) $$A(t,x)>0 \quad \text{in } U.$$

Then there exists an open neighborhood U' of the origin such that any $u\in C^\infty(U)$ satisfying

$$Qu=0, \quad \partial_t^j u|_{t=0}=0 \quad (0\leqq j\leqq p-1),$$

vanishes in U'.

For general p, q and r, the sufficiency of the condition $k\leqq(pr+qm)/(q-r)$ has been studied by Dr. H. Uryu.

Finally we consider non-uniqueness in Gevrey classes. We set $\gamma^{(s)}=C^\infty(\boldsymbol{R}_t;\mathcal{E}^{\{s\}}(\boldsymbol{R}_x))$. Let P be the operator in $\boldsymbol{R}^2$:

$$P=\partial_t^p+t^kA(t)D_x^q-t^mB(t)D_x^{q-r},$$

where p, q, r, k and m are as above, $A, B\in C^\infty(\boldsymbol{R})$, $A(t), B(t)>0$.

Theorem 4. *We assume* (A.1) *and we set*

$$s_0=\frac{p(k-m)}{k(q-r)-pr-qm}.$$

Then, for any $s>s_0$, there exist u and f belonging to $\gamma^{(s)}$ and $\gamma^{(s+1)}$ respectively satisfying

$$Pu-fu=0, \qquad (0,0)\in \operatorname{supp} u\subset\{t\geqq 0\}.$$

Remark 3. When $p=q=2$, $r=1$ and $k=2l$, we have $s_0=(2l-m)/(l-1-m)$. This fact corresponds to the results of Igari [2] and Ivrii [3] on the well-posedness of the Cauchy problem in Gevrey classes.

The method of proofs of Theorems 1 and 2 is a modification of that used in Alinhac and Zuily [1]. That is, we construct u and f by using the method of geometrical optics. We treat the case of higher multiplicity and we take into account of the degeneracy of lower order terms on the initial surfaces. Then, new difficulties arise. In order to prove Theorem 3, we show Carleman type estimates of the form:

$$N^{2p-1}\iint t^{-2N-n-2p}|v|^2\,dtdx\leqq C'\iint t^{-2N-n}|Qv|^2\,dtdx,$$

where $p=3$ or 4, n is an appropriate constant, N is sufficiently large and $v\in C_0^\infty([0,T]\times[-r,r])$. We prove Theorem 4 in the same way as in the proof of Theorem 3 of Nakane [5]. Detailed proofs will be published elsewhere.

§2. Remarks on assumptions (A.2) and (A.3). Assumptions (A.2) and (A.3) imply $A_q(0,0;\xi^0)$, $B_{q-r}(0,0;\xi^0)\neq 0$. Hence we have only to consider the case when there exists $\xi^0\in \boldsymbol{R}^d\backslash\{0\}$ satisfying $A_q(0,0;\xi^0)$, $B_{q-r}(0,0;\xi^0)\neq 0$. Furthermore, by the similarity transformation: $x\to hx$ for some $h>0$, we have the following.

(1) The case when $A_q(0,0;\xi^0)/B_{q-r}(0,0;\xi^0)$ is real. $(1)_1$ When $p\geqq 3$, (A.2) and (A.3) are satisfied if r is odd, or r is even and $A_q(0,0;\xi^0)/B_{q-r}(0,0;\xi^0)>0$. $(1)_2$ When $p=q=2$ and $r=1$, (A.2) and (A.3) are satisfied unless $A_2(0,0;\xi^0)$ is a negative real number.

On the other hand, Watanabe [8] showed that, when $d=1$, $p=q=2$, $r=1$ and $A_2(t,x;\xi)<0$ in $U\times(\boldsymbol{R}\backslash\{0\})$, uniqueness holds for any m and B. Hence assumptions (A.2) and (A.3) are indispensable.

(2) The case when $p=q=2$, $r=1$ and $A_2(0,0;\xi^0)$ is real. (A.2) and (A.3) are satisfied if $A_2(0,0;\xi^0)>0$ and $\operatorname{Re} B_1(0,0;\xi^0)$, $\operatorname{Im} B_1(0,0;\xi^0)\neq 0$.

(3) The case when $p=q=2$, $r=1$ and $B_1(0,0;\xi^0)$ is real. (A.2) and (A.3) are satisfied if $\operatorname{Re} A_2(0,0;\xi^0)>\sqrt{3}\,|\operatorname{Im} A_2(0,0;\xi^0)|$.

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