116. On Compact Kähler Manifolds of Constant Scalar Curvatures

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1. Introduction. The purpose of this note is to generalize the result obtained in [1]. To be more precise we shall present an obstruction to the existence of a Kähler metric of constant scalar curvature in any fixed Kähler class of a compact complex manifold M with $b_1(M)=0$.

Let $\omega=(i/2\pi)\ g_{\alpha\bar\beta}dz^\alpha\wedge d\bar z^\beta$ be a Kähler form of M, $\gamma_\omega=-(i/2\pi)\ \partial\bar\partial\log\det(g_{\alpha\bar\beta})$ the Ricci form of ω, and τ_ω the harmonic part of $\gamma_\omega-\omega$. Then there exists a real valued smooth function F_ω, uniquely determined up to any constant function, such that

$$\gamma_\omega=\omega+\tau_\omega+\frac{i}{2\pi}\partial\bar\partial F_\omega.$$

We denote by $\mathfrak{h}(M)$ the complex Lie algebra of all holomorphic vector fields of M. We define a linear function $f_{[\omega]}$ of $\mathfrak{h}(M)$ into $\boldsymbol{C}$ by

$$f_{[\omega]}(X)=\int_M XF_\omega\omega^m$$

where $m=\dim M$.

Theorem 1. *Let M be a compact complex manifold with $b_1(M)=0$ admitting Kählerian structures. Then the function $f_{[\omega]}$ depends only on the Kähler class $[\omega]\in H^2(M;\boldsymbol{R})$. If M admits a Kähler form $\tilde\omega\in[\omega]$ of constant scalar curvature, then $f_{[\omega]}=0$.*

Theorem 2. *The function $f_{[\omega]}$ is invariant under the group G of all holomorphic transformations of M preserving the class $[\omega]$. In particular the derived algebra of $\mathfrak{h}(M)$ is contained in the kernel of $f_{[\omega]}$ and $f_{[\omega]}$ is a Lie algebra homomorphism. If $\mathfrak{h}(M)$ is semisimple then $f_{[\omega]}=0$. If $f_{[\omega]}\neq 0$ then $\mathfrak{h}(M)$ contains a hyperplane invariant under G.*

If the first Chern class $c_1(M)$ is positive, any Kähler form of constant scalar curvature in $c_1(M)$ is Einstein. So the result of this paper generalizes that of [1].

We remark that $f_{[\omega]}$ actually varies as ω does; this can be observed by considering a product of two compact Kähler manifolds M_i with the Kähler form ω_i, $i=1,2$, such that $f_{[\omega_i]}\neq 0$ and taking Kähler forms $\omega=k_1\omega_1+k_2\omega_2$ on $M_1\times M_2$ for positive parameters k_1 and k_2.

2. Proof. Fix a Kähler form ω_0. Any Kähler form ω_1 cohomologous to ω_0 can be joined by a smooth family of Kähler forms $\omega_t=\omega_0$

$+t(i/2\pi)\,\partial\bar\partial\phi$, ϕ being a real valued smooth function of M. There exists a smooth family θ_t of smooth functions of M, uniquely determined up to any M-constant functions of t, such that

$$\frac{d}{dt}(\tau_{\omega_t})=\frac{i}{2\pi}\partial\bar\partial\theta_t.$$

From now on we shall omit the suffix t for the notational convenience.

Lemma 3. *$\triangle\theta-g^{\alpha\bar\beta}g^{\gamma\bar\delta}\phi_{\alpha\bar\delta}\tau_{\gamma\bar\beta}=0$ where $\tau_\omega=(i/2\pi)\,\tau_{\alpha\bar\beta}dz^\alpha\wedge d\bar z^\beta$.*

Proof. Differentiating the equation $\delta''\tau_\omega=0$ with respect to t, we obtain

$$\bar\partial(\triangle\theta-g^{\alpha\bar\beta}g^{\gamma\bar\delta}\phi_{\alpha\bar\delta}\tau_{\gamma\bar\beta})=0.$$

Since $\delta''\tau_\omega=0$ also implies that

$$\int_M g^{\alpha\bar\beta}g^{\gamma\bar\delta}\phi_{\alpha\bar\delta}\tau_{\gamma\bar\beta}\omega^m=0,$$

the proof of the lemma is immediate.

Lemma 4. *If $b_1(M)=0$, then for each $X\in\mathfrak{h}(M)$ there exists a smooth function ψ such that $X=\psi^\alpha(\partial/\partial z^\alpha)$.*

Proof. Let η be the (0, 1)-form dual to X. Then $\bar\partial\eta=0$. Since $b_1(M)=0$ there exists a smooth function ψ such that $\eta=\bar\partial\psi$, which is the desired function.

Proof of Theorem 1. To prove the first part we need only to show that the derivative of the function $\int_M XF_{\omega_t}\omega_t^m$ of t vanishes identically. In fact one can calculate as in [1]

$$\begin{aligned}\frac{d}{dt}\int_M XF_\omega\omega^m&=\int_M X^\gamma(g^{\alpha\bar\delta}\phi_\alpha\tau_{\gamma\bar\beta}-\theta_\gamma)\omega^m\\&=\int_M \psi(\triangle\theta-g^{\alpha\bar\beta}g^{\gamma\bar\delta}\phi_{\alpha\bar\delta}\tau_{\gamma\bar\beta})\omega^m\\&=0.\end{aligned}$$

The proof of the second part of Theorem 1 is immediate from the fact that harmonicity of τ_ω implies that $g^{\alpha\bar\beta}\tau_{\alpha\bar\beta}$ is constant.

The proof of Theorem 2 is quite similar to that of Theorem 2.1 in [1], and is omitted.

References

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