111. *Signature of Quaternionic Kaehler Manifolds**)

By Tadashi NAGANO**) and Masaru TAKEUCHI***)

(Communicated by Shokichi IYANAGA, M. J. A., Oct. 12, 1983)

We announce that the signature (or the index) of a compact quaternionic Kaehler manifold M equals the Betti number $b_{2n}(M)$, $\dim M = 4n$, and state properties of its cohomology (Theorem 2.6).

1. Facts from Salamon's work. (1.1) Definition. A manifold with Riemannian metric (M, g) is called *quaternionic Kaehlerian* iff the linear holonomy group Ψ_x is contained in $Sp(n)\cdot Sp(1)\subset O(TM_x, g_x)$ for every point x of M, $\dim M=4n$, $n\geqq 2$.

Corresponding to the Lie algebra of $Sp(1)$, there is a parallel vector subbundle V of End $(TM)=TM\otimes T^*M$. (V is a coefficient bundle of imaginary quaternions in [4].) Let Z denote the submanifold of V which consists of the members J satisfying $J^2=-$(the identity map of TM_x), $x=\pi(J)=\text{proj}\,(J)$. Then Z is a sphere bundle $S^2\longrightarrow Z\overset{\pi}{\longrightarrow} M$. (See [4].)

Now we construct an almost complex structure on Z which is known to be integrable [4]. Observe that Z is a parallel fibre subbundle of V and each fibre S^2 has a natural complex structure. Furthermore the tangent space TZ_J to Z at every point J is the direct sum of the tangent space to the fibre and the horizontal space which is isomorphic by the projection with the tangent space TM_x, $x=\pi(J)$, with the complex structure J. Thus one has a complex structure on TZ_J in the obvious fashion.

(1.2) **Definition.** The complex manifold Z with the projection π is the *twistor space* of the quaternionic Kaehler manifold (M, g).

(1.3) **Hypothesis.** We assume that (M, g) *is a compact connected quaternionic Kaehler manifold of positive scalar curvature.*

(1.4) **Theorem** (Salamon [4]). *Under the hypothesis* (1.3), *the twistor space Z has a unique Kaehler metric such that* (1) *the projection π is a Riemannian submersion,* (2) *the aforementioned horizontal subspace is orthogonal to the fibre, and* (3) *the metric on the fibre of Z is a constant multiple of the metric induced from g on M at every point of Z.*

---

*) Partially supported by the Japan Society for Promotion of Science.

**) Department of Mathematics, Osaka University and University of Notre Dame.

***) Department of Mathematics, College of General Education, Osaka University.

(1.5) **Theorem** (Salamon [4]). *We have the complex cohomology* $H^{p,q}(Z)=0$ *for* $p\neq q$ *if the bidegree is defined by the Kaehlerian structure in* (1.4).

2. The cohomology rings of M and Z. Recall the Gysin sequence of the oriented sphere bundle $S^2\to Z\xrightarrow{\pi}M$ which is exact:

(2.1) $0\longrightarrow H^i(M)\xrightarrow{\pi^*}H^i(Z)\xrightarrow{\lambda}H^{i-2}(M)\longrightarrow 0,\quad i\geqq 0,$

where λ is induced by the integration along the fibre $\alpha\mapsto\lambda(\alpha)$ for forms $\alpha\in A^*(Z)$. In particular the cohomology algebra $H^*(M)$ over the reals may be identified with a subalgebra of $H^*(Z)$. We will use some facts in Kaehlerian geometry. (See [2] or [5].) Let $\omega\in A^2(Z)$ denote the Kaehler form on Z. The exterior multiplication by ω induces a linear map $L: H^*(Z)\to H^*(Z): [\alpha]\mapsto[\omega\wedge\alpha]$.

(2.2) **Lemma.** *L gives a splitting of* (2.1) *up to a positive multiple; in particular*

(2.3) $H^i(Z)=H^i(M)\oplus L(H^{i-2}(M)),\quad i\geqq 0.$

(2.4) **Lemma.** $\lambda(\omega^k\wedge\pi^*\alpha)=0$ *for even nonnegative integer k and every form* $\alpha\in A^*(M)$.

The symmetric bilinear form Q on the homogeneous forms $A^{2p}(Z)$, $0\leqq p\leqq n$, defined by

$$Q(\alpha,\beta)=\int_Z \omega^{2n+1-2p}\wedge\alpha\wedge\beta\qquad\text{for } \alpha,\ \beta\in A^{2p}(Z)$$

induces a symmetric bilinear form on $H^{2p}(Z)$, denoted by the same Q. We observe by (2.2) that the signature of M equals that of Q restricted to $H^{2n}(M)$. In terms of this Q, the subspaces in RHS of (2.3) are orthogonal to each other for even $i=2p\leqq 2n$ by (2.4). Thus it is not hard to see that $H^{2p}(M)$ contains the real primitive classes of degree $2p$ on Z. We can prove the following theorem by this fact, Theorem 1.5 and Kaehlerian geometry.

(2.5) **Theorem.** *Under the hypothesis* (1.3), *the symmetric bilinear form Q is positive definite on* $H^{2n}(M)$, *hence the signature of M is the Betti number* $b_{2n}(M)$.

Finally we wish to state an analogue of Kaehlerian geometry. Obviously a quaternionic Kaehler manifold (M,g) has a canonical parallel 4-form Ω. (See [1] and [3].) The cohomology class $[\Omega]$ is a nonzero scalar multiple of $[\omega]^2$ in $H^*(Z)$. Write L_Ω for L^2.

(2.6) **Theorem.** *Under the hypothesis* (1.3), *one has*

(i) $H^i(M)=0$ *for odd* i;

(ii) $(L_\Omega)^{n-p}: H^{2p}(M)\to H^{4n-2p}(M)$, $0\leqq p\leqq n$, *is an isomorphism; and*

(iii) $H^{2p}(M)=\sum_{0\leqq 2k\leqq p}(L_\Omega)^k P^{2p-4k}(M)$, $0\leqq p\leqq n$, *where* $P^{2q}(M)=\{\alpha\in H^{2q}(M)\,|\,(L_\Omega)^{n-q+1}\alpha=0\}$.

References

[1] E. Bonan: Sur l'algèbre extérieure d'une variété presque hermitienne quaternionique. C. R. Acad. Sc. Paris, **296**, 601–602 (1983).

[2] P. Griffiths and J. Harris: Principles of Algebraic Geometry. John Wiley and Sons (1978).

[3] V. Kraines: Topology of quaternionic manifolds. Trans. Amer. Math. Soc., **122**, 357–367 (1966).

[4] S. Salamon: Quaternionic Kähler manifolds. Invent. math., **67**, 143–171 (1982).

[5] A. Weil: Introduction à l'Etude des Variétés Kähleriennes. Hermann (1958).