98. *The Structure of Serial Rings and Self-Duality*

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The notion of serial rings was introduced by T. Nakayama [6]. A left and right Artinian ring R is called *serial* if Re as well as eR has a unique composition series for any primitive idempotent $e \in R$. The structure of serial rings has been studied by many authors (cf. [1], [3]). The purpose of this note is to give a method for the construction of serial rings in general, and we shall give a necessary and sufficient condition for given two serial rings to be Morita equivalent to each other. Moreover, as one of its applications, we shall prove that a serial ring satisfying a mild condition has a self-duality. Proofs and details will be published elsewhere.

1. Let $b_1, b_2, \cdots, b_n$ be a sequence of positive integers such that $b_i \geqq 2$ for $i=2, 3, \cdots, n$ and $b_{[i+1]} \leqq b_i + 1$ for $i=1, 2, \cdots, n$, where $[k]$ denotes the least positive remainder of k modulo n. For each i, let us put $c_i = (1/n)\{b_i - [b_i]\} + 1$ and $d_i = (1/n)\{b_{[i+1]} - 1 - [b_{[i+1]} - 1]\} + 1$. Let $R_1, R_2, \cdots, R_n$ be local uniserial rings such that $c({}_{R_i}R_i) = c_i$ and $R_i/(J_i)^{d_i} \cong R_{[i+1]}/(J_{[i+1]})^{d_i}$ for all i, where $J_i = \mathrm{Rad}(R_i)$ and $c(M)$ denotes the composition length of a module M. Let $\varphi_i : R_i \to R_{[i+1]}$ be a function and $w_i \in R_i$, $i=1, 2, \cdots, n$. Then the system $\mathfrak{S} = \{n\,; b_i, R_i, w_i, \varphi_i\}$ is called a *serial system* if the following four conditions are satisfied: For each i,

(i) $J_i = R_i w_i = w_i R_i$,

(ii) $\pi_{[i+1]} \circ \varphi_i$ is an onto ring homomorphism where $\pi_{[i+1]} : R_{[i+1]} \to R_{[i+1]}/(J_{[i+1]})^{d_i}$ denotes the canonical ring homomorphism,

(iii) $\varphi_i(w_i) \equiv w_{[i+1]} \pmod{(J_{[i+1]})^{d_i}}$,

(iv) $r_i w_i = w_i \varphi_{[i-1]} \circ \varphi_{[i-2]} \circ \cdots \circ \varphi_i(r_i)$ for all $r_i \in R_i$.

Let R be an indecomposable self-basic serial ring with the radical J. Then we can construct a serial system $\mathfrak{S}_R$ associated to R, which will be called an *invariant system* of R, as follows: Let $Re_1, Re_2, \cdots, Re_n$ be a *Kupisch series* for R, i.e., $1_R = e_1 + e_2 + \cdots + e_n$ is a decomposition of 1_R into a sum of mutually orthogonal primitive idempotents such that $c({}_RRe_i) \geqq 2$ for $i=2, 3, \cdots, n$, $Je_i/J^2e_i \cong Re_{i-1}/Je_{i-1}$ for $i=2, 3, \cdots, n$, and $Je_1/J^2e_1 \cong Re_n/Je_n$ if $Je_1 \neq 0$. Let us put $b_i = c({}_RRe_i)$ and $R_i = e_iRe_i$, $i=1, 2, \cdots, n$. For each i, let y_i be an element in $e_iJe_{[i+1]}$ such that $e_iJe_{[i+1]} = R_iy_i = y_iR_{[i+1]}$, and define a function $\varphi_i : R_i \to R_{[i+1]}$

by the formula $r_i y_i = y_i \varphi_i(r_i)$, $r_i \in R_i$. Moreover, let us put $w_i = y_i y_{[i+1]} \cdots y_{[i-2]} y_{[i-1]}$, $i=1, 2, \cdots, n$. Then the system $\mathfrak{S}_R = \{n\,; b_i, R_i, w_i, \varphi_i\}$ is a serial system.

The first main theorem is stated as follows.

Theorem 1. *Let $\mathfrak{S}$ be a serial system. Then there uniquely exists an indecomposable self-basic serial ring R such that $\mathfrak{S}$ is an invariant system of R.*

Let $\mathfrak{S} = \{n\,; b_i, R_i, w_i, \varphi_i\}$ and $\mathfrak{S}' = \{n'\,; b'_i, R'_i, w'_i, \varphi'_i\}$ be serial systems. We shall say that $\mathfrak{S}$ is *equivalent* to $\mathfrak{S}'$ if the following three conditions are satisfied:

(i) $n = n'$,

(ii) there exists an integer m such that

$$b_{[i-m]} = b'_i \qquad \text{for all } i,$$

(iii) for each i, there exist a unit $u'_{[i+1]} \in R'_{[i+1]}$ and a ring isomorphism $\theta_i : R_{[i-m]} \to R'_i$ such that

$$(u'_{[i+1]})^{-1} \varphi'_i(\theta_i(r_{[i-m]})) u'_{[i+1]}$$
$$\equiv \theta_{[i+1]}(\varphi_{[i-m]}(r_{[i-m]})) \ (\text{mod Rad } (R'_{[i+1]})^{d'_i})$$
$$\text{for all } r_{[i-m]} \in R_{[i-m]},$$
$$\theta_i(w_{[i-m]}) = w'_i \Phi'_{[i+1]i}(u'_{[i+1]}) \Phi'_{[i+2]i}(u'_{[i+2]}) \cdots \Phi'_{[i-1]i}(u'_{[i-1]}) u'_i,$$

where $d'_i = (1/n)\{b'_{[i+1]} - 1 - [b'_{[i+1]} - 1]\} + 1$ and $\Phi'_{ji} = \varphi'_{[i-1]} \circ \varphi'_{[i-2]} \circ \cdots \circ \varphi'_{[j+1]} \circ \varphi'_j$ $(i \neq j)$.

The above relation is an equivalence relation of serial systems, and we have the following theorem.

Theorem 2. *A necessary and sufficient condition for given two indecomposable serial rings to be Morita equivalent to each other is that the invariant systems of their basic rings be equivalent to each other. In particular, there is a bijective correspondence between all Morita equivalence classes of indecomposable serial rings and all equivalence classes of serial systems by $[R] \mapsto [\mathfrak{S}_{R^\circ}]$, where $[R]$ denotes the Morita equivalence class of an indecomposable serial ring R, R° denotes the basic ring of R, and $[\mathfrak{S}_{R^\circ}]$ denotes the equivalence class of $\mathfrak{S}_{R^\circ}$.*

2. As an application of Theorem 2, we have the second main theorem which concerns self-duality of serial rings. (Recall that a left and right Artinian ring R has a *self-duality* if there exists a Morita duality between the category of all finitely generated left R-modules and the category of all finitely generated right R-modules.) Let R be an indecomposable self-basic serial ring and $Re_1, Re_2, \cdots, Re_n$ be a Kupisch series for R. Self-duality of serial rings has been studied by J. K. Haack [2], and he has proved that R has a self-duality if $c({}_RRe_1) < c({}_RRe_2) < \cdots < c({}_RRe_n)$. The following theorem is a generalization of his result.

Theorem 3. *With the above notations, R has a self-duality if* $\sharp\{i \mid c({}_R Re_i) \equiv 1 \pmod{n}\} \leqq 1$.

References

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