96. *Some Dirichlet Series with Coefficients Related to Periods of Automorphic Eigenforms. II*[*)]

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§6. This paper is a direct continuation of [2]. Our primary objective here is to begin a discussion of several applications of the general formalism considered in §§2–5.

§7. We start by deriving some estimates for $F_\mu(\xi\,;S^{\pm 1})$. Cf. Theorem 2. The basic procedure is that of analytic number theory. By examining *appropriate* combinations of the Mellin transforms mentioned in [2, p. 416 (line 5)] and applying (4.1), we quickly establish that

(7.1) $$|F_\mu(\xi\,;S^{\pm 1})|=O(1)e^{(\pi/2+\delta)|t|}$$

for $\xi=\sigma+it$, $|\sigma|\leqq N$, $|t|\geqq 1$, $\delta>0$. The implied constant may depend on N, ϕ, δ. Compare [6, pp. 311, 313] and [15, p. 22 (line 12)]. We (can) now combine a Phragmén-Lindelöf argument with (4.1) and theorem 2(v). Cf. [5, p. 95]. This yields:

Theorem 3. *Given* $0<\varepsilon<1/100$ *and* $N\geqq 3$. *Then*:

$$F_\mu(\xi\,;S^{\pm 1})=O\left[\frac{1}{\varepsilon}|t|^{\max(0,3/2-2\sigma,3/2+\varepsilon-\sigma)}\right]$$

for $\xi=\sigma+it$, $|\sigma|\leqq N$, $|t|\geqq 1$. *The implied constant depends solely on* (Γ, N, S, ϕ).

§8. Take $T\geqq 2x\geqq 2000$ and consider the integral

$$\frac{1}{2\pi i}\int_{\partial R}F_\mu(\xi\,;S)\frac{(2\pi x)^{\xi+1}}{\xi(\xi+1)}d\xi \qquad \text{for } \mu=a,\,b$$

with $R=[-\varepsilon,3/2+\varepsilon]\times[-T,T]$. Cf. [5, p. 31]. The "horizontal" contribution is easily estimated using Theorem 3. The contribution from $\{\sigma=-\varepsilon\}$ is then handled using Theorem 2(v) and [15, p. 62 middle]. A typical *component* here reduces to

$$\int_{1000}^{T}G(t)e^{iF(t)}dt$$

with $G(t)=t^{2\varepsilon-1/2}$ and $F(t)=-2t\ln t+2t+t\ln[\pi^2x|S^{-1}[m_0]|]$. The result in [15] is applied to $[T2^{-k-1},\,T2^{-k}]$ for $k\leqq\log_2 T$. *Each* interval of this type splits into $O(1)$ "admissible" subintervals. We conclude that:

$$\frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT}F_\mu(\xi\,;S)\frac{(2\pi x)^{\xi+1}}{\xi(\xi+1)}d\xi=O\left[\frac{x^{1-\varepsilon}}{\varepsilon}T^{2\varepsilon}\ln T\right].$$

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Once this estimate is obtained, the rest is easy. Compare [3, pp. 103–112].

Set:

$$N_a(x)=\sum_{\substack{\{n_0\}\\ 0<S[n_0]\leqq x}} \frac{E[n_0]}{W[n_0]} \qquad N_b(x)=\sum_{\substack{\{n_0\}\\ -x\leqq S[n_0]<0}} I[n_0]$$

$$N_{a1}(x)=\int_1^x N_a(u)du \qquad N_{b1}(x)=\int_1^x N_b(u)du.$$

Theorem 4. *For $x\geqq 1000$ and $\omega=\delta_{m0}\int_{\mathcal{F}}\phi(z)d\mu(z)$, we have:*

$$N_a(x)=\frac{2}{3}\frac{\omega}{\sqrt{qr}}x^{3/2}+O[x^{3/4}\ln x] \qquad N_{a1}(x)=\frac{4}{15}\frac{\omega}{\sqrt{qr}}x^{5/2}+O[x(\ln x)^2]$$

$$N_b(x)=\frac{2\pi}{3}\frac{\omega}{\sqrt{qr}}x^{3/2}+O[x^{3/4}\ln x] \qquad N_{b1}(x)=\frac{4\pi}{15}\frac{\omega}{\sqrt{qr}}x^{5/2}+O[x(\ln x)^2].$$

The implied constants depend solely on (Γ, S, ϕ). To formulate the S^{-1} analog, replace $(qr)^{1/2}$ by $(qr)^{-1/2}$.

This result is a natural extension of the classical Gauss-Siegel theorem. Cf. [13] and [11, pp. 44–45].

§9. Continuing onward: *note* that the behavior of $\theta_m(z;\tau;S)$ with respect to τ can be studied by imitating the development in [14, pp. 85–88, 113–116]. This type of manipulation has become very common in recent years. Cf. [9, p. 455], [10, p. 338], [12, p. 95]. The *trick* is to examine theta functions with characteristic; viz.

(9.1) $$\theta_m(z;\tau;b;S)=\sum_{n\in Z^3} f[\mathcal{W}(\mathcal{M}_z^{-1})(n+b)]$$

where $f(X)=(\sqrt{q}\,x_2-i\sqrt{r}\,x_3)^R e^{\pi i X^t[uS+ivS_1]X}$ and $b\in Z\times\frac{1}{q}Z\times\frac{1}{r}Z$.

Write:

$G_\theta=\{(\begin{smallmatrix}a&b\\c&d\end{smallmatrix})\in SL(2,Z): ab\equiv cd\equiv O \bmod 2\}$

$G_\theta(2qr)=\{(\begin{smallmatrix}a&b\\c&d\end{smallmatrix})\in G_\theta: c\equiv O \bmod 2qr\}$.

Note that G_θ is just the classical theta group. Cf. [1, p. 17].

One finds that:

(9.2) $$\begin{aligned}&\theta_m(z;\tau+2;S)=\theta_m(z;\tau;S);\\ &\theta_m(z;L\tau;S)=i^N(c\tau+d)^{1/2}(\overline{c\tau+d})^{R+1}\theta_m(z;\tau;S)\end{aligned}$$

for $L=(\begin{smallmatrix}a&b\\c&d\end{smallmatrix})\in G_\theta(2qr)$, $c>0$, where

(9.3) $$i^N=\left(\frac{c}{|d|}\right)\left(\frac{2qr}{|d|}\right)\overline{\eta(d)} \quad\text{and}\quad \eta(d)=\begin{Bmatrix}1 \text{ if } d\equiv 1 \bmod 4\\ -i \text{ if } d\equiv 3 \bmod 4\end{Bmatrix}.$$

The proof uses Poisson summation and well-known properties of Gauss sums.

The analogous result for $\theta_m(z;A\tau;S)$—where A is *any* element of G_θ—will involve the qr (inequivalent) functions (9.1). There is no need to write down an exact expression for the coefficients. Compare [14, pp. 87, 114] and [9, 10, 12, *loc. cit.*].

§10. Let

(10.1) $$\Omega_{\theta}(\tau)=v^{R/2+3/4}\int_{\mathcal{F}}\phi(z)\overline{\theta_m(z;\tau;S)}d\mu(z).$$

Cf. Theorem 1 (after correcting the obvious misprint).

In this section we examine $\Omega_{\theta}(\tau)$ from the point-of-view of [4, chap. 9].

First of all: observe that

(10.2) $$\Omega_{\phi}(\tau)=\omega v^{R/2+3/4}$$
$$+\sum_{\substack{\{n_0\}\\ S[n_0]>0}}(-1)^{m/4}\pi\frac{E[n_0]}{W[n_0]}\Gamma(R+1)2^{-R}t^{R/2}(2\pi t)^{(s-R-1)/2}v^{s/2+1/4}\times\Psi\left[\frac{s+R+1}{2};s+\frac{1}{2};2\pi vt\right]e^{-\pi i\bar{\tau}t}$$
$$+\sum_{\substack{\{n_0\}\\ S[n_0]<0}}\sqrt{\pi}\,I[n_0]|t|^{R/2}(2\pi|t|)^{(s-R-1)/2}v^{s/2+1/4}\times\Psi\left[\frac{s-R}{2};s+\frac{1}{2};2\pi v|t|\right]e^{\pi i\tau|t|}.$$

Note that the individual terms are invariant under $s\longleftrightarrow 1-s$.

By applying [4, pp. 348, 420(19)] we immediately see that $\Omega_{\phi}(\tau)$ satisfies

(10.3) $$\Delta_l f+\mathfrak{s}(1-\mathfrak{s})f=0 \qquad \text{on } H \quad \{\text{cf. }(3.1)\}$$

with $l=1/2+R$ and $\mathfrak{s}=s/2+1/4$. In addition: (9.2) and (9.3) show that

(10.4) $$\Omega_{\phi}(L\tau)=\mathcal{U}(L)\Omega_{\phi}(\tau)j_L(\tau;l) \qquad \text{for } L\in \boldsymbol{G}_{\theta}(2qr).$$

Script $\mathcal{U}$ is used to denote THAT multiplier system of weight l on $\boldsymbol{G}_{\theta}(2qr)$ which satisfies:

(10.5) $$\mathcal{U}[\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)]=1$$
$$\mathcal{U}[\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)]=\left(\frac{c}{|d|}\right)\left(\frac{2qr}{|d|}\right)\eta(d) \quad \text{for} \quad \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in \boldsymbol{G}_{\theta}(2qr), \quad c>0.$$

As usual $j_{\sigma}(z;l)=\exp[ilArg(\gamma z+\delta)]$ for $\sigma=\left(\begin{smallmatrix}\alpha&\beta\\\gamma&\delta\end{smallmatrix}\right)\in SL(2,\boldsymbol{R})$. By employing the equation mentioned in §9 paragraph 4, we can *now* study the behavior of $\Omega_{\phi}(\tau)$ as τ approaches the various cusps of $\boldsymbol{G}_{\theta}(2qr)$.

In this way: we ultimately arrive at

Theorem 5. *Define* $\mathcal{F}[\mathfrak{s}(1-\mathfrak{s}),l,\mathcal{U}]$ *as in* [4, pp. 486–7]. *Compare* [7, p. 297]. *Let* $A_0[\mathfrak{s}(1-\mathfrak{s}),l,\mathcal{U}]$ *be the associated subspace of cusp forms. Then:*

(i) $\Omega_{\phi}(\tau)\in A_0[\mathfrak{s}(1-\mathfrak{s}),l,\mathcal{U}]$ *for* $\omega=0$;

(ii) $\Omega_{\phi}(\tau)\in \mathcal{F}\left[\frac{3}{4}\left(1-\frac{3}{4}\right),\frac{1}{2},\mathcal{U}\right]$ *for* $\omega\neq 0$.

Case (ii) can be pushed a bit further by using the orthogonal decomposition mentioned in [8, pp. 290, 302]. In this regard *see also* the $\boldsymbol{G}_{\theta}(2qr)$ analog of [4, p. 532(line 9)] and [8, p. 305 top].

Theorem 5 is a natural extension of [12, pp. 101, 107]. With regard to the holomorphic case: recall [2, p. 416] and note that

$$
(10.6) \quad \left\{
\begin{aligned}
I(\theta) &= \frac{e^{-iR\theta}(\sin\theta)^R}{(k^{-1}-k)^{R-1}} \int_{z_1}^{Pz_1} F(z)[cz^2+(d-a)z-b]^{R-1}dz \\
I[n_0] &= \frac{(-1)^{m/4}}{(k^{-1}-k)^{R-1}} \int_{z_1}^{Pz_1} F(z)[cz^2+(d-a)z-b]^{R-1}dz
\end{aligned}
\right\}
$$

for $\phi = y^R F(z)$

where $P = Qa(k)Q^{-1}$, R = even and positive, and $z_1 \in H$.

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