95. On Approximation by Integral Müntz Polynomials

By Takeshi KANO
Department of Mathematics, Okayama University

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In 1914 there appeared two independent articles of importance on Weierstrass' approximation theorem. Kakeya [6] considered approximation of a given continuous function $f(x)$ on $[a, b]$ by polynomials with integral coefficients, while Müntz [7] studied the condition on the sequence $\Lambda=\{\lambda_n\}$ $(0=\lambda_0<\lambda_1<\cdots<\lambda_n)$ to approximate $f(x)$ by the "Müntz polynomials"

$$p(x)=\sum_{k=0}^{n} a_k x^{\lambda_k}, \tag{1}$$

where the coefficients a_k's are real.

Kakeya proved that on $[0, 1]$ $f(x)$ is uniformly approximated by integral polynomials iff $f(0)$ and $f(1)$ are both integers, and showed that if $\alpha\geqq 4$, $f(x)$ cannot be uniformly approximated on $[0, \alpha]$ by integral polynomials unless it is such a polynomial.

The necessary and sufficient condition found by Müntz was

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n}=+\infty, \tag{2}$$

which is now usually called "Müntz condition". Their aspects and results have been both unified and extended recently (cf. Ferguson [2] for basic results). One of the fundamental problems is to find conditions to approximate $f(x)$ on $[0, \alpha]$ by integral Müntz polynomials, i.e. $p(x)$ with integer coefficients a_k's.

If we denote by $C_0[0, \alpha]$ the set of all continuous functions $f(x)$ on $[0, \alpha]$ such that $f(m)$ is integer for any integer m in $[0, \alpha]$, then Ferguson and Golitschek [3] proved that when Λ is a sequence of positive integers and $\alpha\leqq 1$, (2) is the necessary and sufficient condition for $f\in C_0[0, \alpha]$ being uniformly approximated by integral Müntz polynomials ([2], Chap. 8). Later Golitschek [4] has succeeded in proving this true for any $\lambda_n\uparrow\infty$. Also Ferguson [1] showed, among other things, that the assertion becomes false if $\alpha>1$.

Now define for the increasing sequence Λ of positive numbers,

$$\underline{D}(\Lambda)=\liminf_{N\to\infty}\frac{N}{\lambda_N},\qquad \bar{D}(\Lambda)=\limsup_{N\to\infty}\frac{N}{\lambda_N},$$

which are called respectively the lower and the upper asymptotic densities of Λ. If $\underline{D}(\Lambda)=\bar{D}(\Lambda)<\infty$, we denote it by

$$D(\Lambda)=\lim_{N\to\infty}\frac{N}{\lambda_N},$$

and call it the *asymptotic density* of Λ.

Then Ferguson's result mentioned above may be stated as follows.

Theorem A. *If Λ is a sequence of positive numbers such that there exists a positive constant c satisfying*

(3) $$\lambda_{k+1}-\lambda_k\geqq c, \qquad (k=1,2,\cdots)$$

and $D(\Lambda)=0$, then $f\in C[0,\alpha]$ with $\alpha>1$ cannot be uniformly approximated by integral Müntz polynomials except when f itself is such a polynomial.

He derived this theorem from

Theorem B. *Let Λ be a sequence of positive numbers and define*

$$M_n[0,\alpha]=\inf_{a_k}\sup_{0\leqq x\leqq\alpha}\left|a_0+\sum_{k=1}^{n-1}a_k x^{\lambda_k}+x^{\lambda_n}\right|.$$

Then if

$$\limsup_{n\to\infty} M_n[0,\alpha]>0,$$

$f\in C[0,\alpha]$ cannot be uniformly approximated by integral Müntz polynomials $p(x)$ except the trivial case as mentioned above.

We shall show in this paper that his argument in fact yields the following results.

Theorem 1. *We may replace in Theorem A the asymptotic density by the lower asymptotic density,* i.e. *it is sufficient to assume $\underline{D}(\Lambda)=0$ there.*

Corollary. *If Λ is an infinite primitive sequence of increasing natural numbers, then $f\in C[0,\alpha]$ with $\alpha>1$ cannot be uniformly approximated by integral Müntz polynomials except the trivial case.*

The *primitive sequence Λ* is such that no element of Λ divides any other, and $\underline{D}(\Lambda)=0$ (cf. [5] Chap. V). Ferguson's result ([1] Corollary 3) concerns the special case $\Lambda=P$, the set of all prime numbers.

Before proving Theorem 1, we give the following theorem from which Theorem 1 is easily derived.

Theorem 2. *Let Λ be the same as in Theorem* 1. *Then*

$$\limsup_{n\to\infty}(M_n[0,1])^{1/\lambda_n}=1.$$

Proof. First we observe that (3) implies

$$\lambda_n-\lambda_k\geqq c(n-k), \qquad (1\leqq k\leqq n-1).$$

Next we may suppose $0<c\leqq 2$ and set for some $d>2/c$ $(\geqq 1)$ $\mu_n=d\lambda_n$, so that

$$\mu_{n+1}-\mu_n=d(\lambda_{n+1}-\lambda_n)\geqq cd=c'>2.$$

Since $M_n[0,1]\leqq 1$ for all n, it suffices to prove $\lim_{n\to\infty}\sup M_n^{1/\lambda_n}\geqq 1$. Then we have (cf. [1] Theorem 2)

$$(M_n[0,1])^{1/\lambda_n}\geqq\left(\frac{\lambda_n}{\lambda_n+1}\cdot\frac{1}{\sqrt{2\lambda_n+1}}\cdot\prod_{k=1}^{n-1}\frac{\lambda_n-\lambda_k}{\lambda_n+\lambda_k+1}\right)^{1/\lambda_n}$$

$$=\left(\frac{\mu_n}{\mu_n+d}\cdot\frac{\sqrt{d}}{\sqrt{2\mu_n+d}}\cdot\prod_{k=1}^{n-1}\frac{\mu_n-\mu_k}{\mu_n+\mu_k+d}\right)^{d/\mu_n}$$
$$\geqq\left(\frac{\mu_n}{\mu_n+d}\cdot\frac{1}{\sqrt{2\mu_n+1}}\right)^{d/\mu_n}\cdot\left(\prod_{k=1}^{n-1}\frac{c'(n-k)}{2\mu_n+d}\right)^{d/\mu_n}.$$

Since

$$\lim_{n\to\infty}\left(\frac{\mu_n}{\mu_n+d}\cdot\frac{1}{\sqrt{2\mu_n+1}}\right)^{1/\mu_n}=1,$$

we have

$$\limsup_{n\to\infty} M_n^{1/\lambda_n}\geqq\limsup_{n\to\infty}\left(\prod_{k=1}^{n-1}\frac{c'(n-k)}{2\mu_n+d}\right)^{d/\mu_n}$$
$$\geqq\limsup_{n\to\infty}\left(\frac{c'}{2\mu_n+d}\right)^{d(n-1)/\mu_n}\cdot\left\{\prod_{k=1}^{n-1}(n-k)\right\}^{d/\mu_n}$$
$$\geqq\limsup_{n\to\infty}\{\mu_n^{-n+1}(n-1)!\}^{d/\mu_n}.$$

Hence by Stirling's formula,

$$\limsup_{n\to\infty} M_n^{1/\lambda_n}\geqq\limsup_{n\to\infty}\{\mu_n^{-n+1}(\sqrt{2\pi}-\varepsilon)n^{n-1/2}e^{-n}\}^{d/\mu_n}$$
$$\geqq\limsup_{n\to\infty}\left\{\left(\frac{n}{\mu_n}\right)^{(n-1)/\mu_n}\cdot e^{-n/\mu_n}\right\}^{d}$$
$$\geqq\limsup_{n\to\infty}\left\{\left(\frac{n}{e\mu_n}\right)^{n/e\mu_n}\right\}^{de(n-1)/n}\cdot e^{-d/\mu_n}=1,$$

for $\liminf_{x\to 0+} x=0$ implies $\limsup_{x\to 0+} x^x=1$.

Proof of Theorem 1. It follows from the transformation $x\to x/\alpha$ that

$$M_n[0,\alpha]=\alpha^{\lambda_n}M_n[0,1].$$

By the preceding theorem, there exist infinitely many n such that

$$(M_n[0,1])^{1/\lambda_n}>1-\varepsilon.$$

Thus for infinitely many n we obtain if $\varepsilon<1-1/\alpha$,

$$M_n[0,\alpha]>\{\alpha(1-\varepsilon)\}^{\lambda_n}>1,$$

which proves Theorem 1 according to Theorem B.

Let $\alpha_0=\alpha_0(\Lambda)$ be the infimum of α such that no element of $C[0,\alpha]$ can be uniformly approximated by integral Müntz polynomials except the trivial case. For example, if $\Lambda=cN$ $(c>0)$, $\alpha_0=4$ (cf. [1] Theorem 3). We shall now prove the following

Theorem 3. *If Λ is a sequence of positive number satisfying* (3) *and $\bar{D}(\Lambda)=\delta$, $0<\delta<\infty$, then we have*

$$\alpha_0\leqq\left(\frac{2e}{c\delta}\right)^{1/c}.$$

It is worth noting that every integer sequence Λ with $\bar{D}(\Lambda)>0$ contains arbitrarily long arithmetic progressions [8]. *We remark that* (3) *implies $c\delta\leqq 1$ and hence*

$$\left(\frac{2e}{c\delta}\right)^{1/c}\geqq(2e)^{1/c}>4^{1/c}.$$

Proof of Theorem 3. As shown in the proof of Theorem 2, we have

$$M_n[0,1] \geqq \frac{\lambda_n}{\lambda_n+1} \cdot \frac{1}{\sqrt{2\lambda_n+1}} \cdot \prod_{k=1}^{n-1} \frac{\lambda_n-\lambda_k}{\lambda_n+\lambda_k+1}$$

$$\geqq \frac{\lambda_n}{\lambda_n+1} \cdot (2\lambda_n+1)^{-n+1/2} \cdot c^{n-1} \cdot (n-1)!.$$

Thus by Stirling's formula,

$$\limsup_{n\to\infty} M_n^{1/n} \geqq \limsup_{n\to\infty} \frac{c}{e}\left(\frac{n}{2\lambda_n+1}\right)^{1-1/2n} = \frac{c\delta}{2e}.$$

Therefore for infinitely many n we have

$$M_n > \left(\frac{c\delta}{2e} - \varepsilon\right)^n.$$

Hence, on account of the fact $\lambda_n \geqq cn + (\lambda_1 - c)$,

$$M_n[0,\alpha] > \alpha^{\lambda_n}\left(\frac{c\delta}{2e} - \varepsilon\right)^n \geqq \left\{\alpha^c\left(\frac{c\delta}{2e} - \varepsilon\right)\right\}^n \cdot \alpha^{(\lambda_1 - c)}.$$

Accordingly, if $\alpha^c > 2e/c\delta$, we obtain

$$\limsup_{n\to\infty} M_n[0,\alpha] > 0,$$

which proves Theorem 3 by virtue of Theorem B.

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