92. *Characterization of Amenable Semigroups with a Unique Invariant Mean**)

By Koukichi SAKAI
Department of Mathematics, Kagoshima University

(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 12, 1983)

§ 1. **Introduction.** The purpose of this note is to characterize the algebraic structure of amenable semigroups with a unique left invariant mean. Let S be an abstract semigroup and $m(S)$ be the Banach space of all bounded real functions on S with the supremum norm. Let ψ be a mean on S, i.e., ψ is a positive linear functional on $m(S)$ such that $\psi(1_S)=1$. Here in general 1_A is the characteristic function of any subset A of S. For any $s \in S$ define a mean $s\psi[\psi s]$ on S by $s\psi(f)=\psi({}_sf)$ $[\psi s(f)=\psi(f_s)]$ $(f \in m(S))$, where ${}_sf(t)=f(st)$ $[f_s(t)=f(ts)]$ $(t \in S)$. ψ is said to be *left* [*right*] *invariant* if $s\psi=\psi[\psi s=\psi]$ for all $s \in S$. By $LM(S)$ $[RM(S)]$ we denote the set of all left [right] invariant means on S. S is called *amenable* if $LM(S) \cap RM(S)$ is nonempty. Especially S is called *extremely amenable* if there exists at least one mean ϕ in $LM(S) \cap RM(S)$ such that $\phi(fg)=\phi(f)\phi(g)$ for any f, g in $m(S)$. S is extremely amenable if and only if it has the property (cf. [3]):

(1.1) *For any a, b in S, there exist some s, t in S such that $as=bs$ and $ta=tb$.*

Suppose now S is amenable. Then S has the following property:

(1.2) *For any a, b in S, $aS \cap bS$ and $Sa \cap Sb$ are nonempty.*

By (1.2) we can introduce a congruence (c) on S defined as follows:

(1.3) *For any a, b in S, $a(c)b$ if and only if $as=bs$ and $ta=tb$ for some s, t in S.*

We denote by S_c the factor semigroup of S mod (c), and by ρ the canonical homomorphism of S onto S_c. We note that S_c is two-sided cancellative. When S_c becomes a group, we denote by K the kernel of ρ. Under these notations the main theorem is stated as follows.

Theorem. *Let S be an amenable semigroup. Then the following assertions are mutually equivalent:*

(A_l) *S has a unique left invariant mean.*

(B) *S_c is a finite group and K has the zero element.*

(C) *S contains a finite two-sided ideal which is also a group.*

(A_r) *S has a unique right invariant mean.*

---

*) Dedicated to Prof. Hisaaki Yoshizawa (Department of Mathematics, Kyoto University) on his 60th birthday.

§ 2. **Proof.** In order to prove the main theorem we begin with some lemmas. Let us define a map ρ^* of $m(S)^*$ to $m(S_c)^*$ by $\rho^*(\phi)(h) = \phi(h \circ \rho)$ ($h \in m(S_c)$) for any $\phi \in m(S)^*$. Since ρ is surjective, the following is an immediate consequence of Theorem 1 in [1, p. 531].

Lemma 1. *ρ^* carries $LM(S)$ [$RM(S)$] onto $LM(S_c)$ [$RM(S_c)$].*

The following two results on the uniqueness of invariant mean are essential in the proof of our theorem.

Lemma 2 (Corollary 2.1 in [2]). *A semigroup with left [right] cancellation has a unique left [right] invariant mean if and only if it is a finite group.*

Lemma 3 (Theorem 1 in [5]). *An extremely amenable semigroup has a unique left [right] invariant mean if and only if it has the zero element.*

Lemma 4. *If S_c becomes a group, then we have:*

(1) *$\rho(a)=\rho(b)$ if and only if $as=bs$ and $ta=tb$ for some s, t in K.*

(2) *When K has the zero element p, $\rho(a)=\rho(b)$ if and only if $ap=bp$ and $pa=pb$. In particular $ap=pa$ for all a in S.*

(3) *K is extremely amenable.*

Proof. The "if" parts in (1) and (2) are obvious. Conversely let $\rho(a)=\rho(b)$ and take u, v in S such that $au=bu$ and $va=vb$ (cf. (1.3)). Moreover since S_c is a group, we can find x, y in S such that $\rho(x)=\rho(u)^{-1}$ and $\rho(y)=\rho(v)^{-1}$. Then putting $s=ux$ and $t=yv$, we see that $s, t \in K$, $as=bs$ and $ta=tb$. Especially if p is the zero element in K, we have $ap=a(sp)=(as)p=(bs)p=bp$ and similarly $pa=pb$. In particular $ap=pap=pa$ for all $a \in S$, because $\rho(a)=\rho(pa)=\rho(ap)$. Thus we have (1) and (2). (3) follows from (1) and (1.1). Q.E.D.

A subset σ of S is called a *ρ-representative* if ρ induces a bijective map of σ onto S_c. The following is an analogy of Theorem 4.4 in [4].

Lemma 5 (Lemma 4 in [6]). *Suppose that S_c is a finite group of order n, and let $\sigma=\{a_1, a_2, \cdots, a_n\}$ be a ρ-representative. Then there is a bijective map $L_\sigma[R_\sigma]$ of $LM(K)$ [$RM(K)$] onto $LM(S)$ [$RM(S)$] defined as follows: for any ϕ in $LM(K)$ [$RM(K)$] and $f \in m(S)$,*

$$(2.1) \qquad L_\sigma(\phi)(f)=\frac{1}{n}\sum_{i=1}^{n}\phi(P({}_{a_i}f)) \qquad \left[R_\sigma(\phi)(f)=\frac{1}{n}\sum_{i=1}^{n}\phi(P(f_{a_i}))\right],$$

where $P(f)$ denotes the restriction of f to K.

Proof of the main theorem. Suppose (A_l). Then $LM(S_c)$ is a singleton by Lemma 1. Since S_c is left cancellative, it follows from Lemma 2 that S_c must be a finite group. Accordingly K is extremely amenable by Lemma 4(3), and $LM(K)$ is also a singleton by Lemma 5. So by virtue of Lemma 3, K has the zero element. Thus (A_l) implies (B). Similarly (A_r) implies (B). Suppose (B) and let p be the zero element in K. Then $Sp=pS$ by Lemma 4(2), so that Sp is a two-sided ideal in S. Moreover in view of Lemma 4(2), we get a bijective map $\tilde{\rho}$ of S_c onto Sp defined by $\tilde{\rho}(g)=ap$ ($g \in S_c$), where $a \in S$ with $\rho(a)=g$. As seen easily, $\tilde{\rho}$ is an isomorphism. Therefore S has a two-sided ideal Sp which is isomorphic to the finite group S_c. Hence (B) implies (C). Finally let $\phi \in LM(S)$ and suppose that S has a finite two-sided ideal J which is a group with the identity p. Then $\phi(1_J)=\phi({}_p1_J)=\phi(1_S)=1$, because J is a right ideal. So ϕ is given in the form:

$$\phi(f)=\sum_{i=1}^{n} \lambda_i f(a_i) \qquad (f \in m(S)),$$

where $J=\{a_1, a_2, \cdots, a_n\}$ and λ_i's are nonnegative numbers with $\sum_{i=1}^{n} \lambda_i=1$. Here we note that $J=Sp$ and $sJ=s(pJ)=(sp)J=J$ for all $s \in S$. From this fact we conclude that $\lambda_1=\lambda_2=\cdots=\lambda_n=1/n$. Consequently $LM(S)$ consists of a unique mean ϕ defined by

$$\text{(2.2)} \quad \phi(f)=\frac{1}{n}\sum_{i=1}^{n} f(a_i) \quad (f \in m(S)).$$

Similarly we see that $RM(S)$ is a singleton of ϕ given by (2.2). Therefore (C) implies both (A_l) and (A_r). Thus the proof is complete.

§3. **Remarks.** 1) Suppose (B) and let p be the zero element in K. Then we see that $LM(K)=RM(K)=\{\delta\}$, where δ is defined by $\delta(h)=h(p)$ ($h \in m(K)$). Consequently it follows from Lemmas 4(2) and 5 that $LM(S)=RM(S)=\{\phi\}$, where ϕ is given in the form:

$$\phi(f)=\frac{1}{n}\sum_{i=1}^{n} f(a_i p)=\frac{1}{n}\sum_{i=1}^{n} f(pa_i) \qquad (f \in m(S)),$$

where n is the order of S_c and $\{a_1, a_2, \cdots, a_n\}$ is a ρ-representative. On the other hand, Sp is a unique finite two-sided ideal in S which is a group. More precisely Sp is a group with the identity p, and is isomorphic to S_c.

2) Suppose that S contains a finite two-sided ideal J which is a group with the identity p. Then by the same way as in [6, Theorem 2], we can prove directly that S_c becomes a finite group isomorphic to J, and that p is the zero element in K.

3) The main theorem remains true for any semigroup S with the property (1.2).

References

[1] Day, M. M.: Amenable semigroups. Illinois J. Math., **1**, 509–544 (1957).

[2] Granirer, E.: On amenable semigroups with a finite-dimensional set of invariant means II. ibid., **7**, 49–58 (1963).

[3] ——: Extremely amenable semigroups. Math. Scand., **17**, 177–197 (1965).

[4] Lau, A. T.: Topological semigroups with invariant means in the convex hull of multiplicative means. Trans. Amer. Math. Soc., **148**, 69–84 (1970).

[5] Sakai, K.: Characterization of extremely amenable semigroups with a unique invariant mean. Proc. Japan Acad., **58**(A), 357–360 (1982).

[6] ——: On left amenable semigroups with a unique left invariant mean (to appear in Sci. Rep. Kagoshima Univ., **32** (1983)).