90. *A Shape of Eigenfunction of the Laplacian under Singular Variation of Domains*

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Recently the author has studied a sharp asymptotic behaviour of eigenvalues of the Laplacian under singular variation of domains. See Ozawa [3]–[6]. See Matsuzawa-Tanno [1], Mazja-Nazarov-Plamenevskii [2], for other related topics. In this note we will give a new formula for eigenfunctions of the Laplacian concerning singular variation of domains.

Let Ω be a bounded domain in $\boldsymbol{R}^3$ with smooth boundary $\partial\Omega=\gamma$. Let w be a fixed point in Ω. Let B_ε be the ball defined by $B_\varepsilon=\{z\in\Omega\,;\,|z-w|<\varepsilon\}$ and let $\Omega_\varepsilon=\Omega\backslash\bar{B}_\varepsilon$. Then, the boundary of Ω_ε consists of γ and ∂B_ε. Let $0<\mu_1(\varepsilon)\leq\mu_2(\varepsilon)\leq\cdots$ be the eigenvalues of the Laplacian in Ω_ε under the Dirichlet condition on $\gamma\cup\partial B_\varepsilon$. Let $0<\mu_1\leq\mu_2\leq\cdots$ be the eigenvalues of the Laplacian in Ω under the Dirichlet condition on γ. We arrange them repeatedly according to their multiplicities. Let $\{\varphi_j(\varepsilon)\}_{j=1}^\infty$ (resp. $\{\varphi_j\}_{j=1}^\infty$) be a complete set of orthonormal basis of $L^2(\Omega_\varepsilon)$ (resp. $L^2(\Omega)$) satisfying $-\Delta(\varphi_j(\varepsilon))(x)=\mu_j(\varepsilon)(\varphi_j(\varepsilon))(x)$, $x\in\Omega_\varepsilon$, $(\varphi_j(\varepsilon))(x)=0$ on $\partial\Omega_\varepsilon$ (resp. $-\Delta\varphi_j(x)=\mu_j\varphi_j(x)$, $x\in\Omega$, $\varphi_j(x)=0$ on γ).

We have the following:

Theorem 1. *Fix j. Suppose that μ_j is a simple eigenvalue. Then, the asymptotic relation*

$$\partial(\varphi_j(\varepsilon))(z)/\partial\nu_z^\varepsilon|_{z\in\partial B_\varepsilon}=-\varphi_j(w)\varepsilon^{-1}+O(\varepsilon^{-1/3}) \tag{1}$$

as ε tends to zero. Here $\partial/\partial\nu_z^\varepsilon$ denotes the derivative along the exterior normal direction with respect to Ω_ε.

Remark. Theorem 1 was conjectured in Ozawa [7].

From now on we give a short sketch of our proof of Theorem. We need some lemmas.

Let F be a set in $\boldsymbol{R}^n$. We put

$$|u|_{0,F}=\sup_{x\in F}|u(x)|$$

$$|u|_{\theta,F}=\sup_{x,y\in F}|u(x)-u(y)|/|x-y|^\theta \qquad (0<\theta<1)$$

$$|u|_{1,F}=\sum_{i=1}^n\sup_{x\in F}|\partial_{x_i}u(x)|$$

$$|u|_{2,F}=\sum_{i,j=1}^n\sup_{x\in F}|\partial_{x_i}\partial_{x_j}u(x)|$$

and

$$|u|_{1+\theta,F}=\sum_{i=1}^{n}|\partial_{x_i}u|_{\theta,F}$$

$$|u|_{2+\theta,F}=\sum_{i,j=1}^{n}|\partial_{x_i}\partial_{x_j}u|_{\theta,F} \qquad (0<\theta<1).$$

We have the following

Lemma 1. *Assume that $n\geq 3$. Then, there exists constant C_n independent of ε such that*

$$|u_\varepsilon|_{t,\Omega_\varepsilon}\leq C_n\varepsilon^{-t}\sum_{s=0,1,2,2+\theta}\varepsilon^s|u_\varepsilon|_{s,\partial F_\varepsilon}$$

$(t=0,1,2)$ holds, if u_ε is harmonic in Ω_ε and zero on γ.

Let $G_\varepsilon(x,y)$ (resp. $G(x,y)$) denote the Green function of the Laplacian in Ω_ε (resp. Ω) satisfying

$$\begin{aligned} &-\Delta_x G_\varepsilon(x,y)=\delta(x-y) && x,y\in\Omega_\varepsilon\\ &G_\varepsilon(x,y)=0 && x\in\partial\Omega_\varepsilon,\ y\in\Omega_\varepsilon\\ (\text{resp. } &-\Delta_x G(x,y)=\delta(x-y) && x,y\in\Omega\\ &G(x,y)=0 && x\in\gamma,\ y\in\Omega). \end{aligned}$$

By $\boldsymbol{G}_\varepsilon$ (resp. $\boldsymbol{G}$) we denote the integral operator given by

$$(\boldsymbol{G}_\varepsilon f)(x)=\int_{\Omega_\varepsilon}G_\varepsilon(x,y)f(y)dy$$

$$(\text{resp. } (\boldsymbol{G}g)(x)=\int_{\Omega}G(x,y)g(y)dy).$$

Hereafter we assume that $n=3$. We introduce the following integral operator

$$(\boldsymbol{H}_\varepsilon f)(x)=\int_{\Omega_\varepsilon}h_\varepsilon(x,y)f(y)dy,$$

where

$$h_\varepsilon(x,y)=G(x,y)-4\pi\varepsilon G(x,w)G(y,w)\tau_\varepsilon(x)\tau_\varepsilon(y).$$

Here $\tau_\varepsilon\in C^\infty(\bar{\Omega})$ is a function satisfying $\tau_\varepsilon(x)=1$ on $\Omega_{\varepsilon/2}$, $\tau_\varepsilon(x)=0$ on $\bar{B}_{\varepsilon/4}$, $|\tau_\varepsilon(x)|\leq 1$, $|\mathrm{grad}\,\tau_\varepsilon(x)|\leq 5\varepsilon^{-1}$.

Put $\boldsymbol{Q}_\varepsilon=\boldsymbol{G}_\varepsilon-\boldsymbol{H}_\varepsilon$.

We have the following:

Lemma 2. *Suppose that $u_\varepsilon\in C^\infty(\Omega_\varepsilon)$ satisfies $u_\varepsilon=0$ on $\partial\Omega_\varepsilon$. Let $u_\varepsilon^*(x)\in L^\infty(\Omega_\varepsilon)$ be an extension of u_ε such that $u_\varepsilon^*(x)=0$ for $x\in\bar{B}_\varepsilon$. Then, there exists a constant C_q independent of ε such that*

$$(2)\qquad |\boldsymbol{Q}_\varepsilon u_\varepsilon|_{t,\Omega_\varepsilon}\leq C_q\varepsilon^{-t}(\varepsilon\|u_\varepsilon\|_{L^q(\Omega_\varepsilon)}+\varepsilon^2|\boldsymbol{G}u_\varepsilon^*|_{2,\partial B_\varepsilon}+\varepsilon^{2+\theta}|\boldsymbol{G}u_\varepsilon^*|_{2+\theta,\partial B_\varepsilon}) \qquad (t=1,2)$$

holds, where q being a fixed constant satisfying $q>3$ and $\theta\in(0,1)$.

We consider the following equations:

$$(3)\qquad \begin{aligned}&(\boldsymbol{G}-\mu_j^{-1})\xi_\varepsilon(x)=(\boldsymbol{G}(\tau_\varepsilon\varphi_j))(w)G(x,w)\tau_\varepsilon(x)-(\boldsymbol{G}(\tau_\varepsilon\varphi_j))(w)^2\varphi_j(x)\\ &\int_\Omega \xi_\varepsilon(x)\varphi_j(x)dx=0.\end{aligned}$$

It is easy to see that (3) have the unique solution $\xi_\varepsilon(x)$ in $L^2(\Omega)$. Put

$$(4)\qquad \tilde{\varphi}_j(\varepsilon)=\varphi_j+4\pi\varepsilon\xi_\varepsilon.$$

Let $\kappa_\varepsilon(x)$ be the function satisfying $\kappa_\varepsilon(x)=0$ on $B_{(3/2)\varepsilon}$, $\kappa_\varepsilon(x)=1$ on $\bar{\Omega}_{2\varepsilon}$,

$|\kappa_\varepsilon|\leq 1$, $|\mathrm{grad}\,\kappa_\varepsilon(x)|\leq 3\varepsilon^{-1}$.

We put $\kappa_\varepsilon\tilde{\varphi}_j(\varepsilon)-\varphi_j(\varepsilon)$ in the place of u_ε in (2). Thus, we get

$$(5)\quad \begin{aligned} &|\boldsymbol{G}_\varepsilon\varphi_j(\varepsilon)-\boldsymbol{H}_\varepsilon(\kappa_\varepsilon\tilde{\varphi}_j(\varepsilon))|_{1,\Omega_\varepsilon} \\ &\leq |\boldsymbol{Q}_\varepsilon(\kappa_\varepsilon\tilde{\varphi}_j(\varepsilon))|_{1,\Omega_\varepsilon}+|\boldsymbol{Q}_\varepsilon\varphi_j(\varepsilon)|_{1,\Omega_\varepsilon}+|\boldsymbol{H}_\varepsilon(\kappa_\varepsilon\tilde{\varphi}_j(\varepsilon)-\varphi_j(\varepsilon))|_{1,\Omega_\varepsilon} \\ &\leq C\{\|\kappa_\varepsilon\tilde{\varphi}_j(\varepsilon)\|_{L^q(\Omega_\varepsilon)}+\|\varphi_j(\varepsilon)\|_{L^q(\Omega_\varepsilon)}+\varepsilon(\|\kappa_\varepsilon\tilde{\varphi}_j(\varepsilon)\|_{\theta,\Omega}+\|\varphi_j(\varepsilon)^*\|_{\theta,\Omega})\} \\ &\quad +|\boldsymbol{H}_\varepsilon(\kappa_\varepsilon\tilde{\varphi}_j(\varepsilon)-\varphi_j(\varepsilon))|_{1,\Omega_\varepsilon} \qquad (q>3). \end{aligned}$$

By using (5) we can get our theorem. The estimate for $\|\kappa_\varepsilon\tilde{\varphi}_j(\varepsilon)-\varphi_j(\varepsilon)\|_{L^2(\Omega_\varepsilon)}$ is a crucial fact to estimate $|\boldsymbol{H}_\varepsilon(\kappa_\varepsilon\tilde{\varphi}_j(\varepsilon)-\varphi_j(\varepsilon))|_{1,\Omega_\varepsilon}$. Along this line we get Theorem.

Details of this paper will be given elsewhere.

Added in proof. See also the work of "C. A. Swanson, Cand. Math. Bull., vol. 6, 15–25 (1963)" in which the results concerning singular variation of domain were given.

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