89. *Fourier-Mehler Transforms of Generalized Brownian Functionals**)

By Hui-Hsiung KUO
Department of Mathematics, Louisiana State University

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1. **Generalized Brownian functionals.** In the continuous embeddings $\mathcal{S}\subset L^2(\boldsymbol{R})\subset\mathcal{S}^*$, $\mathcal{S}$ and $\mathcal{S}^*$ are the nuclear spaces of rapidly decreasing functions and tempered distributions, respectively. Let μ be the white noise measure on $\mathcal{S}^*$, i.e. its characteristic functional is given by

$$\int_{\mathcal{S}^*} \exp[i\langle \dot{B}, \xi\rangle]d\mu(\dot{B}) = \exp[-\|\xi\|^2/2] \equiv C(\xi), \qquad \xi\in\mathcal{S},$$

where $\|\cdot\|$ is the $L^2(\boldsymbol{R})$-norm. Being motivated by the well-known Wiener-Ito decomposition of $L^2(\mathcal{S}^*)$, Hida [1], [3] has introduced the following space $(L^2)^-$ of generalized Brownian functionals:

$$(L^2)^- = \sum_{n=0}^{\infty} \oplus K_n^{(-n)},$$

where $K_n^{(-n)}$ consists of generalized multiple Wiener integrals [2]. An element φ in $K_n^{(-n)}$ is realized as a distribution on $\boldsymbol{R}^n$ through the integral transform $\mathcal{T}$:

$$\begin{aligned}(\mathcal{T}\varphi)(\xi) &= \int_{\mathcal{S}^*} \exp[i\langle \dot{B}, \xi\rangle]\varphi(\dot{B})d\mu(\dot{B})\\ &= i^n C(\xi)\int_{\boldsymbol{R}^n} f(u_1,\cdots,u_n)\xi(u_1)\cdots\xi(u_n)du_1\cdots du_n, \qquad \xi\in\mathcal{S},\end{aligned}$$

where f is in the Sobolev space $\hat{H}^{-(n+1)/2}(\boldsymbol{R}^n)$.

2. **Renormalization.** Let T be a finite interval in $\boldsymbol{R}$. By using the renormalization procedure, we obtain the following three generalized Brownian functionals:

1) $\varphi(\dot{B}) = :\exp\left[\lambda\dot{B}(t) + c\int_T \dot{B}(u)^2du\right]:$, $t\in T$, $\lambda, c\in\boldsymbol{C}$, $c\neq 1/2$. The $\mathcal{T}$-transform of φ is given by

$$(\mathcal{T}\varphi)(\xi) = C(\xi)\exp\left[\frac{i\lambda}{1-2c}\xi(t) + \frac{c}{2c-1}\int_T \xi(u)^2du\right], \qquad \xi\in\mathcal{S}.$$

2) $\psi(\dot{B}) = :H_n\left(\dot{B}(t); \frac{1}{(1-2c)dt}\right)\exp\left[c\int_T \dot{B}(u)^2du\right]:$, $c\in\boldsymbol{C}$, $c\neq 1/2$.

The $\mathcal{T}$-transform of ψ is given by

$$(\mathcal{T}\psi)(\xi) = \frac{1}{n!}C(\xi)\left(\frac{i\xi(t)}{1-2c}\right)^n\exp\left[\frac{c}{2c-1}\int_T \xi(u)^2du\right], \qquad \xi\in\mathcal{S}.$$

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3) $\sigma(\dot{b},\dot{B})=\,:\exp\left[\beta\int_T \dot{b}(u)\dot{B}(u)du+c\int_T \dot{b}(u)^2du\right]:_{\dot{b}}$, the renormalization with respect to $\dot{b}$-variable, $\beta, c\in \boldsymbol{C}$, $c\neq 1/2$. The $\mathcal{T}_{\dot{b}}$-transform of σ is given by

$$(\mathcal{T}_{\dot{b}}\sigma)(\eta)=C(\eta)\exp\left[\frac{i\beta}{1-2c}\int_T \dot{B}(u)\eta(u)du+\frac{c}{2c-1}\int_T \eta(u)^2du\right],\qquad \eta\in\mathcal{S}.$$

3. Main results. Theorem 1 (Generating function).

$$:\exp\left[\lambda\dot{B}(t)+c\int_T \dot{B}(u)^2du\right]:=\sum_{n=0}^{\infty}\lambda^n\,:H_n\left(\dot{B}(t);\frac{1}{(1-2c)dt}\right)$$
$$\times\exp\left[c\int_T \dot{B}(u)^2du\right]:,\ t\in T,\ \lambda,\ c\in\boldsymbol{C},\ c\neq 1/2.$$

Let K_τ denote the kernel function

$$K_\tau(\dot{b},\dot{B})=\,:\exp\left[\frac{i}{\sin\tau}\int_T \dot{b}(u)\dot{B}(u)du-\frac{i}{2\tan\tau}\int_T \dot{b}(u)^2du\right]:_{\dot{b}},\qquad \tau\in\boldsymbol{R}.$$

For φ in $(L^2)^-$, we define the Fourier-Mehler transform $\mathcal{G}_\tau\varphi$ of φ by:

$$(\mathcal{G}_\tau\varphi)(\dot{b})=\int_{S^*}K_\tau(\dot{b},\dot{B})\varphi(\dot{B})d\mu(\dot{B}).$$

The Fourier-Mehler transform $\mathcal{G}_\tau$ is a map from $(L^2)^-$ into itself. When $\tau=3\pi/2$, it is the Fourier transform introduced in [4, p. 423]. When $\tau=\pi/2$, it is the inverse Fourier transform.

Theorem 2. *Let* $\Phi(\dot{B})=\,:\exp\left[\lambda\dot{B}(t)-\frac{1}{2}\int_T \dot{B}(u)^2du\right]:,\ t\in T,\ \lambda\in\boldsymbol{R}.$ *Then its Fourier-Mehler transform is*

$$(\mathcal{G}_\tau\Phi)(\dot{b})=\,:\exp\left[\lambda e^{i\tau}\dot{b}(t)-\frac{1}{2}\int_T \dot{b}(u)^2du\right]:.$$

Theorem 3. *Let* $\Phi_n(\dot{B})=\,:H_n\left(\dot{B}(t);\frac{1}{2dt}\right)\exp\left[-\frac{1}{2}\int_T \dot{B}(u)^2du\right]:,$ $t\in T$. *Then its Fourier-Mehler transform is*

$$(\mathcal{G}_\tau\Phi_n)(\dot{b})=e^{in\tau}\,:H_n\left(\dot{b}(t);\frac{1}{2dt}\right)\exp\left[-\frac{1}{2}\int_T \dot{b}(u)^2du\right]:.$$

It follows from Theorem 3 that $\{\mathcal{G}_\tau;\tau\in\boldsymbol{R}\}$ is a one-parameter group acting on the space $(L^2)^-$ of generalized Brownian functionals. Moreover, we can define an arbitrary power of the Fourier transform $\wedge$ [4] by $(\wedge)^r=\mathcal{G}_{3\pi r/2}$.

4. Remarks. Consider the one-dimensional Fourier transform from $L^2(\boldsymbol{R})$ into itself. Let $h_n(x)$ be the normalized Hermite function of degree n. Then its Fourier transform is given by $\hat{h}_n=e^{3\pi in/2}h_n$. This is the motivation for N. Wiener to define the Fourier-Mehler transform $\mathcal{G}_\theta$ [7; 3, p. 260] such that $(\mathcal{G}_\theta h_n)(y)=e^{in\theta}h_n(y)$.

In Hida's theory of generalized Brownian functionals, $\{\dot{B}(t);t\in T\}$ is often regarded as a continuum coordinate system in order to take time propagation into account. Theorem 3 shows that the generalized Brownian functionals

$$\psi_n \equiv :H_n\left(\dot{B}(t);\frac{1}{2dt}\right)\exp\left[-\frac{1}{2}\int_T \dot{B}(u)^2du\right]:,$$

$n=0,1,2,\cdots$, are the infinite dimensional analogues of the Hermite functions. Moreover, $\{\psi_n;\, n\geq 0\}$ forms a basis of the subspace of $(L^2)^-$ spanned by what P. Lévy [6] called the normal functionals. More generally, $d\nu=\exp\left[c\int \dot{B}(u)^2du\right]d\mu$, $c\neq 1/2$, can be regarded formally as a Gaussian measure on $\mathcal{S}^*$ with variance $1/(1-2c)$. Then $\dot{B}(t)$ with respect to $d\nu$ is a Gaussian random variable with variance $1/(1-2c)dt$. Therefore,

$$:H_n\left(\dot{B}(t);\frac{1}{(1-2c)dt}\right)\exp\left[c\int B(u)^2du\right]:$$

is the Hermite function with respect to $d\nu$.

The detailed proofs of the above results and other formulas concerning Fourier and Fourier-Mehler transforms will appear in [5].
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