86. *Free Arrangements of Hyperplanes over an Arbitrary Field**)

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(Communicated by Kunihiko KODAIRA, M. J. A., Sept. 12, 1983)

In [6], we proved a factorization theorem for the Poincaré polynomial of the complement of hyperplanes in an l-dimensional vector space over the complex number field $\boldsymbol{C}$ when the arrangement of the hyperplanes is free. That was called Shephard-Todd-Brieskorn theorem there. Our main aim here is to report a generalized factorization theorem for a free arrangement over an arbitrary field. The detailed proof will appear in [3].

1. Let A be an arrangement in an l-dimensional vector space V over a field K. In other words, A is a finite family of $(l-1)$-dimensional vector subspaces of V. Denote the dual vector space of V by V^*. Let $S=S(V^*)$ be the symmetric algebra of V^*. Fix a base $\{x_1, \cdots, x_l\}$ for V^*, and S is isomorphic to the polynomial algebra $K[x_1, \cdots, x_l]$. Let $Q \in S$ be a reduced defining equation for $\bigcup_{H\in A} H$. Then Q is a product of elements of V^*. The derivation of S is a K-linear map $\theta: S \to S$ satisfying $\theta|_K \equiv 0$ and $\theta(fg)=f\theta(g)+g\theta(f)$ for any $f, g \in S$.

Definition 1. *A derivation along* A (which is called a logarithmic vector field [4] when we are in the complex analytic category) is a derivation θ of S satisfying

$$\theta(Q) \in QS.$$

Let $D(A)$ denote the set of derivations along A. Then $D(A)$ is naturally an S-module.

Definition 2. If $D(A)$ is an S-free module, we say that A is *a free arrangement.*

Definition 3. A derivation θ of S is said to be *homogeneous of degree* b if $\theta(x_i) \in S_b$ $(i=1, \cdots, l)$, where S_b is the vector subspace of S generated by monomials of degree b. We write $b=\deg \theta$. We can show that $D(A)$ has a free base $\{\theta_1, \cdots, \theta_l\}$ consisting of homogeneous derivations if A is a free arrangement. The integers $(\deg \theta_1, \cdots, \deg \theta_l)$ are called *the degree* of A (called the generalized exponents of A in [6]). They depend only upon A.

The following useful criterion, proved by K. Saito [4] when $K=\boldsymbol{C}$, remains true for arbitrary K:

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*) Partially supported by JMS research fellowship.

Proposition 1. *For homogeneous* $\theta_1, \cdots, \theta_l \in D(A)$, 1) $\theta_1 \wedge \cdots \wedge \theta_l =: \det [\theta_i(x_j)]_{1 \le i,j \le l} \in QS$, 2) $\theta_1, \cdots, \theta_l$ *are a free base for* $D(A)$ *if and only if* $\theta_1 \wedge \cdots \wedge \theta_l \in K^*Q$ ($K^* = K \setminus \{0\}$).

2. We will define combinatorial notions. Let $L(A) = \{\bigcap_{H \in B} H\,;\, B \subseteq A\}$. (Agree that $\bigcap_{H \in \phi} H = V$.) Introduce a partial order $\ge$ by $X \ge Y$ iff $X \subseteq Y$. Then V is the minimal element. We simply write L instead of $L(A)$.

Definition 4. The *Möbius function* μ on L is inductively defined by

$$\mu(V) = 1,$$
$$\mu(Y) = -\sum_{\substack{X < Y \\ X \in L}} \mu(X) \qquad (Y \in L).$$

The *characteristic polynomial* $\chi(A, t) \in Q[t]$ for an arrangement A is defined by

$$\chi(A, t) = \sum_{X \in L} \mu(X) t^{\dim X}.$$

In [1], Orlik-Solomon showed that $(-t)^l \chi(A, t^{-1})$ equals the Poincaré polynomial $\sum_{i \ge 0} \dim H^i(M) t^i$ ($M = V \setminus \bigcup_{H \in A} H$) when $K = \boldsymbol{C}$. Our main result is

Factorization theorem (see [6] when $K = \boldsymbol{C}$). *For a free arrangement A with its degrees* $(b_1, \cdots, b_l)$,

$$\chi(A, t) = \prod_{i=1}^{l} (t - b_i).$$

Example 1. Let $K = \boldsymbol{C}$. When A is the set of all reflecting hyperplanes of a finite unitary reflection group (over $\boldsymbol{C}$), A is free. In this case, Factorization theorem was first proved by Orlik-Solomon [2].

Example 2. Let $K = \boldsymbol{F}_q$ (a field with q elements). Let A be the arrangement consisting of all $(l-1)$-dimensional subspaces of V. Define

$$\theta_i = \sum_{j=1}^{l} x_j^{q^{i-1}} (\partial / \partial x_j) \qquad (i = 1, \cdots, l).$$

Let $\alpha = \sum_{j=1}^{l} c_j x_j \in V^*$. Then

$$\theta_i(\alpha) = \sum_{j=1}^{l} x_j^{q^{i-1}} c_j$$
$$= \left(\sum_{j=1}^{l} c_j x_j\right)^{q^{i-1}} \in \alpha S.$$

For each $H \in A$, fix an element $\alpha_H \in V^*$ such that $H = \ker(\alpha_H)$. Note that $\theta_i(\alpha_H) \in \alpha_H S$ ($H \in A$) by the argument above. Let $Q = \prod_{H \in A} \alpha_H$.

$$\theta_i(Q) = \sum_{H \in A} (Q / \alpha_H) \theta_i(\alpha_H) \in QS.$$

Thus $\theta_1, \cdots, \theta_l \in D(A)$. The determinant

$$\theta_1 \wedge \cdots \wedge \theta_l = \begin{vmatrix} x_1 & \cdots x_l \\ x_1^q & \cdots x_l^q \\ \vdots & \vdots \\ x_1^{q^{l-1}} & \cdots x_l^{q^{l-1}} \end{vmatrix}$$

is not zero because the coefficient of $x_1 x_2^q \cdots x_l^{q^{l-1}}$ is 1. One also has

$$\sum_{i=1}^{l} \deg \theta_i = 1 + q + \cdots + q^{l-1}$$
$$= (q^l - 1)/(q - 1) = \sharp A = \deg Q.$$

Thanks to Proposition 1, these imply that $\theta_1 \wedge \cdots \wedge \theta_l \in K^*Q$ and that $\theta_1, \cdots, \theta_l$ are a free base for $D(A)$. Thus A is free. In this case, by Factorization theorem, one has

$$\chi(A, t)=(t-1)(t-q)\cdots(t-q^{l-1}).$$

3. Fix $H_0 \in A$. Define

$$A'=A\setminus\{H_0\},$$
$$A''=\{H \cap H_0,\ H \in A'\}.$$

The arrangement A'' is an arrangement in the $(l-1)$-dimensional vector space H_0.

Addition-deletion theorem. *Any two of the following three conditions imply the other one.*

(1) *A is free with its degrees $(b_1, \cdots, b_l)$,*

(2) *A' is free with its degrees $(b_1, \cdots, b_{l-1}, b_l-1)$,*

(3) *A'' is free with its degrees $(b_1, \cdots, b_{l-1})$.*

This was proved in [5] for $K=\boldsymbol{R}$ or $\boldsymbol{C}$. The principle of our proofs for Factorization theorem and Addition-deletion theorem is essentially same as our proofs when $K=\boldsymbol{C}$ [6] [5]. In order to overcome the obstruction which appears when K is a finite field, the following lemma is crucial:

Lemma 1 (Stability of freeness under an algebraic field extension). *Let A be an arrangement over K. Suppose that F is an algebraic field extension of K. Denote the corresponding arrangement in $V \otimes_K F$ by A_F. Then the arrangement A_F over F is free if and only if the arrangement A over K is free.*

By using Lemma 1, we can prove the following proposition, which is important in our proofs for the two theorems above, for an arbitrary field K:

Proposition 2. *If A is free, then so is $A_X = :\{H \in A\,;\, H \supseteq X\}$ for any $X \in L$.*

Acknowledgment. This work was done during my stay at University of Wisconsin at Madison in 1982–83. I would like to express special thanks to Profs. P. Orlik and L. Solomon who made the stay possible and gave valuable suggestions.

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