85. A Type of Comparison Theorem in Polynomial Growth Cohomology

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This note is a continuation of our previous works [6]–[8], and we point out here that the cohomology group of algebraic coherent sheaves over a quasi-projective variety is expressed in terms of polynomial growth cohomology and a method of stratification theory (Theorems 1, 2). This may be regarded as a generalization of the Gaga comparison in Serre [10] and its generalizations to locally free sheaves over a smooth quasi-projective variety (Cornalba-Griffiths [2] and Deligne-Maltsiniotis [5]). The content here does not contain essentially new substantial facts from those in [6]. But it may be an indispensable task to make clear how the arguments in [6] are related to the gaga comparison in [10]. Also methods here (applied also to some other analytic varieties) will provide a theoretical base for some recent explicit computations on holomorphic bundles [9]. (Because this note requires some subtle facts, which were not given explicitly in [6], we will write a detail elsewhere.)

1. Polynomial growth (p.g.) variety. We begin this note by refining some notions in § 1, [6]: Let X be a topological space: By a growth function of X we simply mean a map $g: X \to \boldsymbol{R}_{+1} := [1, \infty)$. Next letting Y be a subset of X, take a growth function g_Y of Y and a distance function d_X of X. Setting $\mathfrak{g}_Y = (g_Y, d_X)$ and taking an element $\sigma = (\sigma_1, \sigma_2) \in \boldsymbol{R}^2_{+1}$ we define the following neighborhood of a point p $(\in Y)$ in X:

(1.1) $$N(X; p; \mathfrak{g}_Y)_\sigma := \{q \in X; d_X(q, p) < (1/\sigma \cdot g_Y(p))\},$$

where, as in [6], for an element $r \in \boldsymbol{R}_+$, we set $\sigma \cdot r = \sigma_1 \cdot r^{\sigma_2}$.

Definition 1. (1) By *σ-$\mathfrak{g}_Y$-covering* (resp. *σ-$\mathfrak{g}_Y$-neighborhood*) *of Y in X*, we mean:

(1.2) $$\begin{cases} \mathcal{A}(X; Y; \mathfrak{g}_Y)_\sigma := \{N(X; p; \mathfrak{g}_Y)_\sigma; p \in Y\} \\ N(X; Y; \mathfrak{g}_Y)_\sigma := \bigcup_{p \in Y} N(X; p; \mathfrak{g}_Y)_\sigma. \end{cases}$$

(2) Assuming that $Y = X$, we mean by *$\mathfrak{g}_X$-uniform structure of X* the following assignment:

(1.3) $$\mathcal{A}: \boldsymbol{R}^2_{+1} \ni \sigma \longrightarrow \mathrm{Cov}\,(X) \ni \mathcal{A}(X; X; \mathfrak{g}_X)_\sigma.$$

The p.g. neighborhood in (1.2) is convenient for investigations of analytic subvarieties. The p.g. uniform structure in (1.2) is suitable for our Cech arguments ([6]). Thirdly, for an abelian sheaf $\mathfrak{F}$ over

X, we have 'q-structure' $|\ |_{\mathfrak{F}}$ of $\mathfrak{F}$. (This is an obvious abstraction of 'absolute value' for continuous functions; cf. §1, [6].)

Now let $\boldsymbol{P}^N$ be the projective space of dimension N and d the distance function of it determined by the Fubini metric. Let X be a quasi projective subvariety of $\boldsymbol{P}^N$ and $\bar{X}$ the completion of it in $\boldsymbol{P}^N$. Taking a real analytic function f_X on $\bar{X}$, whose locus is $\bar{X}-X$, we take $g_X=(g_X, d_X)$, $g_X=1/f_X$, to be 'standard p.g. structure' of X. Next let $\mathfrak{F}$ be an algebraic coherent sheaf over X. Then it is well known that $\mathfrak{F}$ is factored as: $\mathfrak{E}\xrightarrow{\omega}\mathfrak{F}\to 0$, where $\mathfrak{E}$ is the restriction to X of a locally free sheaf $\bar{\mathfrak{E}}$ over $\bar{X}$. Taking a fiber metric of the corresponding bundle $\bar{E}$, and we set $|\ |_{\mathfrak{F}}=\omega_*(|\ |_{\mathfrak{E}})$, where $|\ |_{\mathfrak{E}}$ is the q-structure of $\mathfrak{E}$ induced from the metric and $|\ |_{\mathfrak{F}}$ is the induced q-structure from it by ω (p. 380, [6]). We take this to be 'standard q-structure' of $\mathfrak{F}$. (The choice of the function f_X and the factorization: $\mathfrak{E}\xrightarrow{\omega}\mathfrak{F}\to 0$ as well as the fiber metric do not influence on the definition of the cohomology groups soon below.)

2. Cohomology groups. Letting $(X, \mathfrak{F})$ be just as above, take a closed subvariety V of X. Then we have the p.g. neighborhood $N(V)_\sigma =N(X; V; g_V)_\sigma$, $g_V=(g_{X|V}, d_X)$ of V in X(§1). Letting $\mathfrak{N}(V)$ denote the direct system $\{N(V)_\sigma; \sigma\in R^2_{+1}\}$, we will attach to $(\mathfrak{N}(V), \mathfrak{F})$ two types of analytic cohomology groups.

2.1. P.g. complex. First letting $g_\sigma:=(g_{X|N(V)_\sigma}, d_X)$ we have a p.g. covering $\mathcal{A}(X; N(V)_\sigma)_\sigma$ of $N(V)_\sigma$ in X. Writing this covering as $\mathcal{A}_\sigma$, we have the following complex (cf. Definitions 1.9 and 2.6 in [6]):

$$C^{\cdot}_{p.g}(X; \mathfrak{N}(V), \mathfrak{F}):=\lim_{\sigma\to} C^{\cdot}_{p.g}(\mathcal{A}_\sigma, \mathfrak{F}), \tag{2.1}$$

where

$$C^{\cdot}_{p.g}(\mathcal{A}_\sigma, \mathfrak{F}):=\{\varphi\in C^{\cdot}(\mathcal{A}_\sigma, \mathfrak{F});\ \varphi \text{ satisfies the p.g. condition: } |\varphi(p)| <\alpha\cdot g_X(p) \text{ in } |\mathcal{B}| \text{ for each element } \mathcal{B} \text{ of the nerve } Nv^{\cdot}(\mathcal{A}_\sigma)\}. \tag{2.1$'$}$$

(Here the element $\alpha\in \boldsymbol{R}^2_{+1}$ is independent of $\mathcal{B}$ and p.)

2.2. A stratification method. Now take a stratification $\mathcal{S}$ of V, whose strata are quasi projective subvarieties of V. We then define the nerve $Nv^{\cdot}_{\mathcal{S}}=\coprod_{q\geq 0} Nv^q_{\mathcal{S}}$ of $\mathcal{S}$ to be: $Nv^q_{\mathcal{S}}=\{\mathcal{U}=(S_{\lambda_0}, \cdots, S_{\lambda_q})\}$ where the elements $S_{\lambda_i}\in\mathcal{S}$ satisfy: $S_{\lambda_0}\prec\cdots\prec S_{\lambda_q}$. (Here $S_{\lambda_0}\prec S_{\lambda_1}, \cdots$ means that $S_{\lambda_0}\subset\bar{S}_{\lambda_1}, \cdots$). We then define the following 'p.g. neighborhood' of $\mathcal{U}$:

$$N(X; \mathcal{U})_\sigma:=\textstyle\bigcap_{j=0}^q N(X; S_{\lambda_j}; g_{\lambda_j}), \tag{2.2}$$

where the p.g. structure $g_{S_{\lambda_j}}$ of S_{λ_j} is defined to be: $(g_{S_{\lambda_j}}, d_X)$, with a p.g. function $g_{S_{\lambda_j}}$ of S_{λ_j}. It is convenient to define the p.g. function of $\mathcal{U}$ to be $g_{S_{\lambda_q}}$=that of the highest term S_{λ_q} (cf. [8]), and for the direct system $\mathfrak{N}(\mathcal{U}):=\{N(X; \mathcal{U})_\sigma; \sigma\in \boldsymbol{R}^2_{+1}\}$, we define the following p.g. group:

(2.3) $$\Gamma_{p\cdot g}(X;\mathcal{N}(\mathcal{U}),\mathfrak{F}):=\lim_{\sigma\to}\Gamma_{p\cdot g}(N(X;\mathcal{U})_\sigma,\mathfrak{F}),$$

where

(2.3)′ $\Gamma_{p\cdot g}(X;\mathcal{N}(\mathcal{U}),\mathfrak{F}):=\{\varphi\in\Gamma(N(X;\mathcal{U})_\sigma,\mathfrak{F}),\ |\varphi(p)|<\alpha\cdot g_{S_{\lambda_q}}(p)$ in $N(X;\mathcal{U})_\sigma$, with suitable σ, $\alpha\in \boldsymbol{R}^2_{+1}\}$.

We regard $Nv^{\cdot}_{\mathcal{S}}$ as the category by the order: $\mathcal{U}\succ\mathcal{U}'\Leftrightarrow\mathcal{U}\subset\mathcal{U}'$, and we have a contravariant functor:

(2.4) $C_{\mathcal{S}'\mathfrak{F}}: Nv^{\cdot}_{\mathcal{S}}\ni\mathcal{U}\longrightarrow\{\text{abelian groups}\}\ni C_{\mathcal{S}'\mathfrak{F}}(\mathcal{U})=\Gamma_{p\cdot g}(X;\mathcal{N}(\mathcal{U}),\mathfrak{F})$.

By a standard argument we have the following complex from $C_{\mathcal{S}'\mathfrak{F}}$:

(2.5) $$C^{\cdot}_{p\cdot g}(X;\mathcal{N}(\mathcal{S}),\mathfrak{F}):=\oplus_{q\geq 0}\oplus_{\mathcal{U}\in Nv^q_{\mathcal{S}}}C_{\mathcal{S}'\mathfrak{F}}(\mathcal{U}),$$

where $\mathcal{N}(\mathcal{S})$ denotes the collection $\{\mathcal{N}(X;S_\lambda);\ S_\lambda\in\mathcal{S}\}$. For two stratifications $\mathcal{S},\mathcal{S}'$ of V, we write $\mathcal{S}\succ\mathcal{S}'$, if there is a refining map: $\mathcal{S}'\supset\mathcal{S}$. We then have the following complex, which we may call 'stratification complex' for $(\mathcal{N}(V),\mathfrak{F})$:

(2.6) $$C^{\cdot}_{str}(X;\mathcal{N}(V),\mathfrak{F}):=\lim_{\mathcal{S}\to}C^{\cdot}_{p\cdot g}(X;\mathcal{N}(\mathcal{S}),\mathfrak{F}).$$

We write the cohomology groups of the complexes in (2.1) and (2.6) as $H^{\cdot}_{p\cdot g}(\cdots)$ and $H^{\cdot}_{str}(\cdots)$. Then we have:

Theorem 1. $H^{\cdot}_{str}(X;\mathcal{N}(V),\mathfrak{F})\simeq H^{\cdot}_{p\cdot g}(X;\mathcal{N}(V),\mathfrak{F})$.

Assume that $V=X$, and we write the above groups as $H^{\cdot}_{str}(X,\mathfrak{F})$ and $H^{\cdot}_{p\cdot g}(X,\mathfrak{F})$. Then, for the algebraic cohomology group $H^{\cdot}_{alg}(X,\mathfrak{F})$, we have:

Theorem 2. *We have the following isomorphism*:

(2.7) $$\begin{array}{ccc} H^{\cdot}_{alg}(X,\mathfrak{F}) & \xrightarrow{\sim} & H^{\cdot}_{str}(X,\mathfrak{F}) \\ & \searrow\wr \quad \wr\swarrow & \\ & H^{\cdot}_{p\cdot g}(X,\mathfrak{F}) & \end{array}$$

When X is a projective variety, the isomorphism: $H^{\cdot}_{alg}\simeq H^{\cdot}_{p\cdot g}$ coincides with the original Gaga comparison in Serre [10]:

(2.8) $$H^{\cdot}_{alg}(X,\mathfrak{F})\xrightarrow{\sim}H^{\cdot}_{an}(X,\mathfrak{F}).$$

(Also see the generalizations in [2] and [5], which use the $\bar\partial$-estimations. See also [6] for the isomorphism: $H^{\cdot}_{alg}\simeq H^{\cdot}_{p\cdot g}$ for the affine map situation, which depends on a certain Cech argument.)

The proof of Theorems 1, 2 will be given elsewhere. Here we summarize what we feel the key point in the proof: Letting $\mathcal{S}$ and $N(\mathcal{U})_\sigma:=N(X;\mathcal{U})_\sigma$ be as in (2.3), we have a p.g. covering $\mathcal{A}(X;\mathcal{U})_\sigma:=\mathcal{A}(X;N(\mathcal{U})_\sigma;g_{S_{\lambda_t}})$ of $N(\mathcal{U})_\sigma$ in X(§1) and a Cech complex:

(3.1) $C^{\cdot}_{p\cdot g}(X,\mathcal{N}(\mathcal{U}),\mathfrak{F}):=\lim_{\sigma\to}C^{\cdot}_{p\cdot g}(\mathcal{A}(X;\mathcal{U})_\sigma,\mathfrak{F})$,

where the right hand side $=\{\varphi\in C^{\cdot}(\mathcal{A}(X;\mathcal{U})_\sigma,\mathfrak{F});\ |\varphi(p)|<\alpha\cdot g_{S_{\lambda_t}}(p)\}$ in $|\mathcal{B}|$ for each $\mathcal{B}\in Nv^{\cdot}(\mathcal{A}(X;\mathcal{U})_\sigma)$ and $p\in|\mathcal{B}|$, with a suitable $\alpha\in R^2_{+1}$. Then the following double complex plays key roles in the proof.

(3.2) $K=\bigoplus_{p,q\geq 0}K^{p,q}$, with $K^{p,q}=\bigoplus_{\mathcal{U}\in Nv^p_{\mathcal{S}}}C^q_{p\cdot g}(X,\mathcal{N}(\mathcal{U}),\mathfrak{F})$.

Actually letting d_1 and d_2 denote the degree one maps: $K^{p,q}$

$\rightarrow K^{p+1,q}$, $K^{p,q+1}$, and letting denote by E'_r and E''_r the corresponding spectral sequences, we easily have:

(3.3) $\quad E_1'^{\cdot,0} \cong C^{\cdot}_{p.q}(X, \mathcal{N}(\mathcal{S}), \mathfrak{F})$ and $E_1''^{\cdot,0} \cong C^{\cdot}_{p.q}(X, \mathcal{N}(V), \mathfrak{F})$.

But a very simple observation implies: $E_1''^{q,\cdot} \cong 0 (q \geqq 1)$, while we reduce $E_1'^{q,\cdot} (\cong H^q_{p.q}(X, \mathcal{N}(\mathcal{U}), \mathfrak{F})) \cong 0 (q \geqq 1)$ to 'Thereom B type results' in [6] (cf. in particular, § 4.1). These facts and (3.3) will lead, in the standard manner, to Theorems 1 and 2.

Remark. Some stratification methods' approaches to holomorphic bundles, where we focus growth properties of certain transition matrices in question with respect to codimension two subvarieties, will be found in [9].

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