78. On the Borel Summability of $\sum_{n=1}^{\infty} n^{-\alpha}\exp(in^{\beta}\theta)$

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0. In the previous paper [1] we proved that $\sum_{n=1}^{\infty} n^{-\alpha}\sin(n^{\beta}\theta)$ is convergent if $1<\beta<2\alpha$. We now consider the Borel summability and apply a Tauberian theorem (cf. [2] Theorem 156) to show that this series converges for all $\theta>0$ when $\alpha=1/2$, $\beta<5/4$.

In fact we prove

Theorem (cf. [2] Notes, Chapter IX). *If $\beta<\alpha+3/4$ then*

$$\sum_{n=1}^{\infty} n^{-\alpha}\exp(in^{\beta}\theta) \qquad (i^2=-1) \tag{1}$$

is Borel summable for all $\theta>0$.

Corollary. *If $\alpha\geqq 1/2$ and $1<\beta<5/4$, then* (1) *converges for all $\theta>0$.*

1. **Main lemma.** For $1<\beta<3/2$, $\theta>0$ and positive integers m, set

$$F(t)=\left(\frac{\beta}{2}\right)\theta m^{\beta-2}t^2+\beta\theta m^{\beta-1}t=At^2+Bt, \quad \text{say.} \tag{2}$$

Let μ be such that as $m\to\infty$,

$$\mu/m^{1/2+\delta}\longrightarrow 1 \qquad \left(0<\delta<\frac{1}{2}-\frac{\beta}{3}\right). \tag{3}$$

Lemma.

$$\int_{-\mu}^{\mu}\frac{t}{m}\exp\left(-\frac{t^2}{2m}\right)\sin(F(t)-2k\pi t)dt=O\left(\frac{m^{-1/4+(3/2)\delta}}{\sqrt{|k-B/2\pi|}}\right), \tag{4}$$

where k is a positive integer, and $B/2\pi\notin Z$.

Proof. We write

$$(4)=m^{-1}\left(\int_{-\mu}^{0}+\int_{0}^{\mu}\right)=m^{-1}(J_1+J_2), \quad \text{say.}$$

We only consider J_2 here since J_1 similarly estimated.

By changing variable and by the second mean value theorem, we have

$$J_2=\frac{1}{2m}\int_0^{\xi}\sin(Au+(B-2k\pi)\sqrt{u})du.$$

Then by van der Corput's lemma (cf. [3] Lemma 4.4), we have

$$J_3=O\left(m^{-1/4+(3/2)\delta}\Big/\sqrt{\left|k-\frac{B}{2\pi}\right|}\right), \qquad \text{where } \frac{B}{2\pi}\notin Z.$$

2. Sketch of proof of Theorem. We take Borel's integral method, that is, we consider

$$\int_0^\infty e^{-x}\sum_{n=1}^{\infty} n^{-\alpha}\exp(in^{\beta}\theta)\frac{x^n}{n!}\,dx.$$

It suffices for the convergence to show that as $x\to+\infty$

$$e^{-x}\sum_{n=1}^{\infty} n^{-\alpha}\exp(in^{\beta}\theta)\frac{x^n}{n!}=O(x^{-\nu}), \tag{5}$$

for some constant $\nu>1$. Following the argument in [2] (Theorem 137 (9.1.6)), we consider

$$e^{-x}\sum_{r=-\mu}^{\mu}(m+r)^{-\alpha}\exp(i(m+r)^{\beta}\theta)\frac{x^{m+r}}{(m+r)!}. \tag{6}$$

By the substitution $(m+r)^{-\alpha}=m^{-\alpha}(1-\alpha(r/m)+O(r^2/m^2))$ and by a variant of Theorem 137 (9.1.8) of [2], we may obtain

$$(6)=\frac{1}{\sqrt{2\pi}}m^{-\alpha-1/2}\sum_{r=-\mu}^{\mu}\left(1+C_1\frac{r}{m}+C_2\frac{r^3}{m^2}+O\left(\frac{r^4}{m^3}+\frac{1}{m}\right)\right)$$
$$\times\exp\left(-\frac{r^2}{2m}+i(m+r)^{\beta}\theta\right),$$

where C_1 and C_2 are bounded with respect to both m and r.

Since $m^{\beta-3}r^3$ is bounded because of (3), if we expand $\exp(i(m+r)^{\beta}\theta)$ $=\exp(im^{\beta}(1+r/m)^{\beta}\theta)$, then we see that the main term in the expansion is

$$m^{-\alpha-1/2}\exp(im^{\beta}\theta)\sum_{r=-\mu}^{\mu}\exp\left(-\frac{r^2}{2m}+iF(r)\right). \tag{7}$$

By the Euler summation formula,

$$(7)=m^{-\alpha-1/2}\exp(im^{\beta}\theta)\cdot\left[\int_{-\mu}^{\mu}\exp\left(-\frac{t^2}{2m}+iF(t)\right)dt\right.$$
$$+\int_{-\mu}^{\mu}\phi(t)\left(\exp\left(-\frac{t^2}{2m}+iF(t)\right)\right)'dt$$
$$\left.+\frac{1}{2}\left(\exp\left(-\frac{\mu^2}{2m}+iF(-\mu)\right)+\exp\left(-\frac{\mu^2}{2m}+iF(\mu)\right)\right)\right]$$
$$=(8)+(9)+\frac{1}{2}[(10)+(11)],\quad\text{say},$$

where $\phi(t)=-\sum_{k=1}^{\infty}\sin(2k\pi t)/k\pi$, and both of (10) and (11) are $O(m^{-\gamma})$ for any $\gamma>1$ due to (3).

Integration by parts then will show that for some $\gamma>1$

$$m^{-\alpha-1/2}\exp(im^{\beta}\theta)\int_{-\mu}^{\mu}\exp\left(-\frac{t^2}{2m}\right)\begin{matrix}\cos\\ \sin\end{matrix}\,F(t)dt=O(m^{-\gamma}). \tag{12}$$

Now we estimate (9). It is sufficient to consider

$$m^{-\alpha-1/2}\exp(im^{\beta}\theta)\int_{-\mu}^{\mu}\phi(t)\left[\exp\left(-\frac{t^2}{2m}\right)\begin{matrix}\cos\\ \sin\end{matrix}\,F(t)\right]'dt. \tag{13}$$

After term by term integration, we can write the integral in (13) as

$$(14)\qquad \sum_{k=1}^{\infty}\frac{1}{k\pi}\int_{-\mu}^{\mu}\sin(2k\pi t)\left[\frac{t}{m}\exp\left(-\frac{t^2}{2m}\right)\begin{matrix}\cos\\ \sin\end{matrix}F(t)\right.$$

$$\left.\pm\exp\left(-\frac{t^2}{2m}\right)F'(t)\begin{matrix}\sin\\ \cos\end{matrix}F(t)\right]dt$$

$$=(15)+(16),\quad\text{say}.$$

$$(15)=\sum_{k=1}^{\infty}\frac{1}{2k\pi}\left[\pm\int_{-\mu}^{\mu}\frac{t}{m}\exp\left(-\frac{t^2}{2m}\right)\begin{matrix}\sin\\ \cos\end{matrix}(F(t)+2k\pi t)dt\right.$$

$$\left.\mp\int_{-\mu}^{\mu}\frac{t}{m}\exp\left(-\frac{t^2}{2m}\right)\begin{matrix}\sin\\ \cos\end{matrix}(F(t)-2k\pi t)dt\right]$$

$$=(17)+(18),\quad\text{say}.$$

By the Lemma, $(18)=O(\sum_{k=1}^{\infty} m^{-1/4+(3/2)\delta}/k\sqrt{|k-B/2\pi|})$, and by integration by parts,

$$(17)=O\left(m^{-1/2}\sum_{k=1}^{\infty}\frac{1}{k^2}\right)=O(m^{-1/2}).$$

Similarly,

$$(16)=O(m^{1-\beta})+O\left(m^{-1/4+(3/2)\delta}\sum_{k=1}^{\infty}\frac{1}{k\sqrt{|k-B/2\pi|}}\right).$$

On the other hand

$$\sum_{k=1}^{\infty}\frac{1}{k\sqrt{|k-B/2\pi|}}=\sum_{1\le k<B/2\pi}\frac{1}{k\sqrt{|k-B/2\pi|}}+\sum_{k>B/2\pi}\frac{1}{k\sqrt{|k-B/2\pi|}}$$

$$=O\left(\log\frac{B}{2\pi}\cdot\left\{\frac{B}{2\pi}\right\}^{-1/2}\right)+O\left(\left(1-\left\{\frac{B}{2\pi}\right\}\right)^{-1/4}\right),$$

where $\{\lambda\}$ denotes the fractional part of λ.

Therefore

$$(13)=O(m^{-\alpha-\beta-1/2})+O\left(m^{-\alpha-3/4+(3/2)\delta}\left\{\frac{\beta\theta}{2\pi}m^{\beta-1}\right\}^{-1/2}\right)$$

$$+O\left(m^{-\alpha-3/4+(3/2)\delta}\left(1-\left\{\frac{\beta\theta}{2\pi}m^{\beta-1}\right\}\right)^{-1/4}\right),$$

$$(19)\qquad \int_K^{\infty}(13)dx=O\left(\int_K^{\infty}x^{-\alpha-\beta+1/2}dx\right)+O\left(\int_K^{\infty}\frac{dx}{x^{\alpha+3/4-(3/2)\delta}\sqrt{\{(\beta\theta/2\pi)x^{\beta-1}\}}}\right)$$

$$+O\left(\int_K^{\infty}\frac{dx}{x^{\alpha+3/4-(3/2)\delta}\sqrt[4]{(1-\{(\beta\theta/2\pi)x^{\beta-1}\})}}\right).$$

Since for any given $0<a<1$ and $b>1$ there exists a constant $K>1$ such that both of

$$\int_K^{\infty}\frac{dx}{\sqrt{\{x^a\}}x^b}\quad\text{and}\quad\int_K^{\infty}\frac{dx}{\sqrt[4]{(1-\{x^a\})}x^b}$$

converge, we know that all the integrals in (19) are convergent for some $K>1$. Hence follows the convergence of (7). In like manners we can estimate all the other remaining terms and finally obtain (5).

Q. E. D.

References

[1] M. Akita and T. Kano: On the Convergence of $\sum_{n=1}^{\infty} n^{-\alpha} \sin(n^{\beta}\theta)$. Proc. Japan Acad., **58A**, 172–174 (1982).
[2] G. H. Hardy: Divergent Series. Oxford (1949).
[3] E. C. Titchmarsh: The Theory of the Riemann Zeta-Function. Oxford (1951)