72. G-Vector Bundles and F-Projective Modules*)

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§1. **Introduction.** Swan has shown that there is a one to one correspondence between vector bundles over a compact Hausdorff space X and finitely generated projective modules over the ring of continuous real-valued functions on X [7].

In the present paper, we will consider an equivariant version of this. Let G be a compact topological group. Then a notion of G-vector bundles is already defined [1]. On the other hand, we introduce notions of equivariant modules, of a family F of equivariant modules and of F-projective modules so that we have an equivariant Swan theorem.

For each family F, we define two kinds of equivariant algebraic K-theories associated with F. Taking a suitable family F, we have an isomorphism of an equivariant topological K-theory and our equivariant algebraic K-theory associated with F.

Equivariant algebraic K-theory is studied along the line of Quillen [6] by Fiedorowicz, Hauschild and May [4], while our approach is along the line of the classical algebraic K-theory [5]. The reason will clear up in a subsequent paper. Namely we will show that our equivariant algebraic K-theory is a Mackey functor [3]. Accordingly the Dress induction theorem [2] is applicable. Using our equivariant Swan theorem, we will show that Brauer and Artin type induction theorems hold in equivariant topological K-theories $KO_G(X)$ and $K_G(X)$. Accordingly equivariant topological K-theories are characterized by the hyperelementary subgroups.

§2. **Families and equivariant algebraic K-theory.** The word *ring* will always mean associative ring with an identity element 1. Let G be a group. A *G-ring* is a ring Λ together with a G-action on Λ preserving the ring structure. If Λ is a G-ring, a *ΛG-module* is a module M over Λ together with a G-action on M such that

$$(*)\qquad g(\lambda_1 m_1+\lambda_2 m_2)=(g\lambda_1)(gm_1)+(g\lambda_2)(gm_2)$$
$$\text{for any } g\in G,\ \lambda_i\in\Lambda,\ m_i\in M.$$

A collection F of finitely generated ΛG-modules is called a *family* if the following holds;

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"if M_1, $M_2 \in F$, then there exists an element $N \in F$ such that $M_1 \oplus M_2$ is a direct summand of N".

When Λ is a commutative G-ring, we can consider a product of two ΛG-modules as follows. If M_1 and M_2 are ΛG-modules, define $M_1 \otimes M_2$ to be $M_1 \otimes_\Lambda M_2$ as a Λ-module with G-action by $g(m_1 \otimes m_2) = gm_1 \otimes gm_2$ for $g \in G$, $m_i \in M_i$.

When Λ is a commutative G-ring, a collection F of finitely generated ΛG-modules is called a *multiplicative family* if in addition to the above condition the following holds;

"if M_1, $M_2 \in F$, then there exists an element $N \in F$ such that $M_1 \otimes M_2$ is a direct summand of N".

Each element of F is called *F-free*. A ΛG-module M is called *F-projective*, if there exists a ΛG-module N so that $M \oplus N$ is F-free.

We introduce two kinds of equivariant algebraic K-groups as follows. $K^G(\Lambda; F)_d$ (resp. $K^G(\Lambda; F)_e$) is defined to be the abelian group given by generators $[P]$ where P is a F-projective ΛG-module, with relations

$$[P] = [P'] + [P'']$$

whenever $P \cong P' \oplus P''$ (resp. $0 \to P' \to P \to P'' \to 0$ is an exact sequence of ΛG-modules).

If Λ is a commutative G-ring and if F is a multiplicative family of ΛG-modules, the product above induces a structure of commutative ring in $K^G(\Lambda; F)_d$ (not in $K^G(\Lambda; F)_e$ in general).

§ 3. **Equivariant Swan theorem.** Let Δ be one of the classical fields $\boldsymbol{R}$ (the real numbers), $\boldsymbol{C}$ (the complex numbers) or $\boldsymbol{Q}$ (the quaternions). Let X be a compact Hausdorff G-space. A ΔG-vector bundle ξ on X is a Δ-vector bundle together with a G-action on ξ preserving the Δ-vector bundle structure [1]. The set of isomorphism classes of ΔG-vector bundles on X forms an abelian semi-group under the Whitney sum. The associated abelian group is denoted by $K\Delta_G(X)$. The tensor product of G-vector bundles induces a structure of commutative ring in $K\Delta_G(X)$ for $\Delta = \boldsymbol{R}$ or $\boldsymbol{C}$.

Let $C_\Delta(X)$ be the ring of continuous Δ-valued functions on X. Then G acts on $C_\Delta(X)$ by $(g \circ a)(x) = a(g^{-1}x)$ for $g \in G$, $a \in C_\Delta(X)$ and $x \in X$. With these definitions, $C_\Delta(X)$ becomes a G-ring. Then Δ is a G-subring of $C_\Delta(X)$ by regarding an element $a \in \Delta$ as the constant function of value a.

The set $\Gamma(\xi)$ of all sections of ξ is a module over $C_\Delta(X)$ and G acts on $\Gamma(\xi)$ by $(g \circ s)(x) = gs(g^{-1}x)$ for $g \in G$, $s \in \Gamma(\xi)$, $x \in X$. It is easy to see that with these definitions $\Gamma(\xi)$ becomes a $C_\Delta(X)G$-module in the sense of § 2. In fact the notions of G-rings and ΛG-modules are abstracted from $C_\Delta(X)$ and $\Gamma(\xi)$.

Let V be a finite dimensional G-representation space over Δ. Regarding $C_\Delta(X)$ as a right Δ-module, we form a finitely generated $C_\Delta(X)G$-module $C_\Delta(X)\otimes_\Delta V$. Let $\boldsymbol{F}_r$ be the set consisting of such modules $C_\Delta(X)\otimes_\Delta V$. Then $\boldsymbol{F}_r$ becomes a family in the sense of §2. If $\Delta=\boldsymbol{R}$ or $\boldsymbol{C}$, then $\boldsymbol{F}_r$ is a multiplicative family. Denote by $\underline{\underline{V}}$ the trivial bundle $p: X\times V\to X$.

Theorem 3.1. *Let G be a compact topological group and X be a compact Hausdorff G-space. Then a $C_\Delta(X)G$-module P is isomorphic to a module of the form $\Gamma(\xi)$ (resp. $\Gamma(\underline{\underline{V}})$) if and only if P is $\boldsymbol{F}_r$-projective (resp. $\boldsymbol{F}_r$-free).*

§4. Twisted group ring $\widetilde{\Lambda G}$. So far we used the term ΛG as an adjective. We now introduce a *twisted group ring* $\widetilde{\Lambda G}$. As an additive group, $\widetilde{\Lambda G}$ is the ordinary group ring and the multiplication is given by

$$(\sum \lambda_g g)\circ(\sum \lambda'_g g')=\sum \lambda_g(g\cdot\lambda'_g)gg'$$

for $g,\ g'\in G,\ \lambda_g,\ \lambda'_g\in\Lambda$. Then $\widetilde{\Lambda G}$ is a ΛG-module in the sense of §2.

Let $\boldsymbol{F}_t$ be the family consisting of the direct sum $(\widetilde{\Lambda G})^n$ of n copies of $\widetilde{\Lambda G}$ where $n=1, 2, 3, \cdots$. If Λ is a commutative G-ring, then $\boldsymbol{F}_t$ is a multiplicative family. Denote by $K_0(\ )$ the ordinary algebraic K_0 group [5].

Theorem 4.1. *We have the following isomorphisms of abelian groups:*

$$K^G(\Lambda;\boldsymbol{F}_t)_d \underset{\text{(I)}}{\cong} K^G(\Lambda;\boldsymbol{F}_t)_e \underset{\text{(II)}}{\cong} K_0(\widetilde{\Lambda G}).$$

If Λ is commutative, (I) *is an isomorphism of rings.*

Proof. Theorem 4.1 is proved by showing that every short exact sequence of $\boldsymbol{F}_t$-projective modules is split exact.

Remark 4.2. Theorem 4.1 implies that our definition of an equivariant algebraic K-group includes $K_0(\widetilde{\Lambda G})$ as a special case. Even if Λ is commutative, $\widetilde{\Lambda G}$ is not commutative in general and $K_0(\widetilde{\Lambda G})$ has no canonical ring structure.

Theorem 4.3. *We have the following isomorphisms of abelian groups:*

$$K\Delta_G(X) \underset{\text{(I)}}{\cong} K^G(C_\Delta(X);\boldsymbol{F}_r)_d \underset{\text{(II)}}{\cong} K^G(C_\Delta(X);\boldsymbol{F}_t)_d$$
$$\underset{\text{(III)}}{\cong} K^G(C_\Delta(X);\boldsymbol{F}_t)_e \underset{\text{(IV)}}{\cong} K_0(\widetilde{C_\Delta(X)G}).$$

If $\Delta=\boldsymbol{R}$ or $\boldsymbol{C}$, then (I)–(III) *are isomorphisms of commutative rings.*

Proof. Since every irreducible representation over Δ is a direct summand of the regular representation, the isomorphism (II) follows. The isomorphisms (I), (III) and (IV) follow from Theorems 3.1 and 4.1.

Remark 4.4. Since $\widetilde{C_\Delta(X)G}$ is not commutative in general,

$K_0(\widetilde{C_\Delta(X)G})$ has no canonical ring structure even if $\Delta=\boldsymbol{R}$ or $\boldsymbol{C}$. Hence $K_0(\widetilde{\Lambda G})$ is insufficient as equivariant algebraic K-theory. This is one of the reasons why we introduced $K^G(\Lambda;F)_d$ and $K^G(\Lambda;F)_e$.

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