71. Classification of Logarithmic Fano 3-Folds

By Hironobu MAEDA
Department of Mathematics, University of Tokyo

(Communicated by Kunihiko KODAIRA, M. J. A., June 14, 1983)

§ 1. **Introduction.** The purpose of this note is to outline our recent results on the structure of logarithmic Fano 3-folds. Details will be published elsewhere. Our proof of the results is based on the theory of threefolds whose canonical bundles are not numerically effective, due to S. Mori [6], and the theory of open algebraic varieties, due to S. Iitaka [2].

Let X be a non-singular projective variety over an algebraically closed field k of characteristic zero. Let $D=D_1+D_2+\cdots+D_s$ be a divisor with simple normal crossings on X.

A pair (X, D) is called a logarithmic Fano variety if $-K_X-D$ is an ample divisor, In the case where $D=0$, X turns out to be a Fano variety in the usual sense.

A logarithmic Fano variety of dimension two may be called a logarithmic del Pezzo surface.

§ 2. **General properties.** Let (X, D) be a logarithmic Fano variety of an arbitrary dimension. By using Norimatsu vanishing theorem [8, Theorem 1], we have the following

Lemma 2.1. (1) $\kappa(X)=-\infty$ *and* $\kappa^{-1}(X)=\dim X$.

(2) $\mathrm{Pic}\,(X)\cong H^2(X, Z)$. *In particular,* $\rho(X)=B_2(X)$.

(3) $\mathrm{Pic}\,(X)$ *is torsion free.*

The boundary D of a logarithmic Fano variety (X, D) satisfies the following

Lemma 2.2. (1) $D_i\cap D_j\neq\phi$ *for any* i *and* j.

(2) $s\leqq\dim X$.

§ 3. **Classification of logarithmic del Pezzo surfaces.**

Lemm 3.1. *Let* (S, Γ) *be a logarithmic del Pezzo surface. Then the* Δ*-genus* [1, Definition 1.4] *of* S *with respect to* $-K_S-\Gamma$ *is as follows:*

(a) *If* $\Gamma=0$, *then* $\Delta(S, -K_S)=1$.

(b) *If* $\Gamma\neq0$, *then* $\Delta(S, -K_S-\Gamma)=0$.

Using the results of T. Fujita [1, pp. 107–110] on polarized varieties of Δ-genera zero, we have the following

Proposition 3.2. *Let* (S, Γ) *be a logarithmic del Pezzo surface. If* $\Gamma\neq0$, *then* (S, Γ) *is one of the following* 7 *pairs:*

(i) $S\cong\boldsymbol{P}^2$, $\Gamma=\Gamma_1$ *where* Γ_1 *is a line.*

(ii) *$S\cong \boldsymbol{P}^2$, $\Gamma=\Gamma_1+\Gamma_2$ where each Γ_i is a line.*

(iii) *$S\cong \boldsymbol{P}^2$, $\Gamma=\Gamma_1$ where Γ_1 is a non-singular conic.*

(iv) *$S\cong \Sigma_n=\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1}\oplus\mathcal{O}_{\boldsymbol{P}^1}(-n))$, $\Gamma=\Gamma_1$ where Γ_1 is a section with $(\Gamma_1)^2_S=-n$.*

(v) *$S\cong \Sigma_n$, $\Gamma=\Gamma_1+\Gamma_2$ where Γ_1 is a section with $(\Gamma_1)^2_S=-n$ and Γ_2 is a fiber.*

(vi) *$S\cong \Sigma_1$, $\Gamma=\Gamma_1$ where Γ_1 is a section with $(\Gamma_1)^2_S=1$.*

(vii) *$S\cong \Sigma_0$, $\Gamma=\Gamma_1$ where Γ_1 is a section with $(\Gamma_1)^2_S=2$.*

§4. Extremal rational curves on logarithmic Fano 3-folds. Let $NE(X)$ be a cone generated by all effective 1-cycles in $N(X)=A^1(X)\otimes_{\boldsymbol{Z}}\boldsymbol{R}$.

For their notations and definitions we refer to [6].

By extended Mori's theory, due to S. Tsunoda [9], $NE(X)$ is a polyhedral cone for a logarithmic Fano variety, i.e.

$$NE(X)=\boldsymbol{R}_+[\ell_1]+\cdots+\boldsymbol{R}_+[\ell_r]$$

where each ℓ_i is a curve such that

$$0<(-K_X-D\cdot\ell_i)\leqq \dim X+1.$$

Lemma 4.1. *Let (V,D) be a logarithmic Fano 3-fold. Then there exists an extremal rational curve ℓ satisfying the following conditions:*

(1) $(D\cdot\ell)>0$.

(2) *The type of ℓ is either* C_2, D_2, D_3, E_2 *or* F *in a sense of S. Mori* ([5] *or* [7]).

§5. Classification of boundaries of logarithmic Fano 3-folds. Let (V,D) be a logarithmic Fano 3-fold with non-zero boundary $D=D_1+\cdots+D_s$. Let $\Gamma_i=D_i|_{D_1}$ for $i\neq 1$. Since

$$(-K_V-D)|_{D_1}=K_{D_1}-\Gamma_2-\cdots-\Gamma_s$$

is an ample divisor on D_1, $(D_1, \Gamma_2+\cdots+\Gamma_s)$ is a logarithmic del Pezzo surface. By the same reason, the $(D_j, (D-D_j)|_{D_j})$ are logarithmic del Pezzo surfaces.

If D consists of only one component, i.e. $D=D_1$, then D is a del Pezzo surface in the usual sense. The configurations of D is determined by Lemma 4.1 and Proposition 3.2.

§6. Classification of logarithmic Fano 3-folds. Fano 3-folds have been classified by V. A. Iskovskih [4], S. Mori and S. Mukai [7]. For a logarithmic Fano 3-fold (V,D) where $D\neq 0$, we obtain the following result.

Theorem. *Let (V,D) be a logarithmic Fano 3-fold with $D\neq 0$. Then (V,D) must be one of 5 types:*

(i) *V is either $\boldsymbol{P}^3$, Q_2, V_1, V_2, V_3, V_4 or V_5 in the notations of Iskovskih* [4]. *Letting H be an ample generator of* Pic(V), *we have $-K_V\sim rH$, where r is the index of V. In this case D is a member of $|tH|$, with $t<r$.*

(ii) *V is a $\boldsymbol{P}^1$-bundle over a non-singular surface which is either a del Pezzo surface or a Hirzebruch surface Σ_n. One of the components of D is a birational section of this bundle and another component, if exists, is formed by fibers.*

(iii) *V is a quadric fibering over $\boldsymbol{P}^1$ with $B_2(V)=2$. V is embedded in a $\boldsymbol{P}^3$-bundle over $\boldsymbol{P}^1$ as an ample divisor. One of the components of D is a horizontal one of this fibering. Another component, if exists, is a fiber.*

(iv) *V is a $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^1$, denoted by Σ_{a_1,a_2}. D has one or two horizontal components. Another component, if exists, is a fiber.*

(v) *V is obtained either from $\boldsymbol{P}^3$ by blowing up non-singular conic or from another logarithmic Fano 3-fold (V', D') by blowing up some points lying on a boundary D'. V' is either $\boldsymbol{P}^3$, Q_2 orΣ_{a_1,a_2}.*

The author would like to thank Prof. S. Iitaka for his helpful suggestions and encouragement.

References

[1] T. Fujita: On the structure of polarized varieties with Δ-genera zero. J. Fac. Sci. Univ. Tokyo, **22**, 103–115 (1975).

[2] S. Iitaka: Algebraic Geometry, an Introduction to Birational Geometry of Algebraic Varieties. GTM, **76**, Springer (1981).

[3] V. A. Iskovskih: Anticanonical models of three dimensional algebraic varieties. J. Soviet Math., **13**, 815–868 (1980).

[4] ——: Three dimensional Fano varieties I, II. Math. USSR Izvestija, **11**, 485–527 (1977); ibid., **12**, 469–509 (1978).

[5] M. Miyanishi: Algebraic methods in the theory of algebraic threefolds. Algebraic Varieties and Analytic Varieties. North Holland-Kinokuniya, pp. 69–99 (1983).

[6] S. Mori: Threefolds whose canonical bundles are not numerically effective. Ann. of Math., **116**, 133–176 (1982).

[7] S. Mori and S. Mukai: On Fano 3-folds with $B_2 \geqq 2$. Algebraic Varieties and Analytic Varieties. North Holland-Kinokuniya, pp. 101–129 (1983).

[8] Y. Norimatsu: Kodaira vanishing theorem and Chern classes for ∂-manifolds. Proc. Japan Acad., **54A**, 107–108 (1978).

[9] S. Tsunoda: Open surfaces of logarithmic Kodaira dimension $-\infty$. KAISEKI TAYŌTAI Seminar, Univ. of Tokyo (1981) (in Japanese).