# 70. *Fourier Coefficients of Generalized Eisenstein Series of Degree Two*

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**Introduction.** In [7] we obtained an explicit formula of Fourier coefficients $a(T, [f])$ of the Eisenstein series $[f]$ of degree two attached to an elliptic eigenform $f$ in case where $-|2T|$ is a fundamental discriminant. In this note we report how the results in [7] extend to the case of general $T$. This is a résumé of [8]. Theorem 1 below is used in [10], where integrality and congruence properties of $a(T, [f])$ are studied. The author would like to thank Profs. H. Maaß and N. Kurokawa for their suggestions and encouragements.

§ 1. **A unified explicit formula of $a(T, [f])$.** We follow the notation of [7] throughout this paper. By [7, Remark 2] we limit our attention to the case where semi-integral $T=\begin{pmatrix} t_1 & t/2 \\ t/2 & t_2 \end{pmatrix}$ is primitive, i.e., g.c.d. $(t_1, t_2, t)=1$. We put $|2T|=\Delta(T)\mathfrak{f}^2$ with a positive integer $\mathfrak{f}$ and a fundamental discriminant $-\Delta(T)$.

**Theorem 1.** *Let $f \in M_k(\Gamma_1)$ be a normalized eigenform where $k$ is a positive integer. Suppose $T>0$ is primitive. Then we have*:

$$(*)\quad a(T, [f])=(-1)^{k/2}\frac{(k-1)!}{(2k-2)!}(2\pi)^{k-1}\Delta(T)^{k-(3/2)}\frac{L(k-1, \chi)}{L_2(2k-2, f)} \cdot\sum_{\substack{s|\mathfrak{f}\\ s>0}} M(\mathfrak{f}s^{-1})\sum_{\substack{t|s\\ t>0}}\mu(t)D(k-1, f, \vartheta_T^{(s/t)}).$$

Here $\vartheta_T^{(v)}=\sum_{n\geq 0} b_T(nv^2)\mathrm{e}(nz)$ if we write $\vartheta_T(z)=\sum_{n\geq 0} b_T(n)\mathrm{e}(nz)$; $\chi$ is the Dirichlet character associated with $\boldsymbol{Q}(\sqrt{-\Delta(T)})$, $\mu$ is the Möbius function, and

$$M(a)=\sum_{\substack{d|a\\ d>0}}\mu(d)\chi(d)d^{k-2}\sigma_{2k-3}(ad^{-1}) \quad \text{where} \quad \sigma_s(a)=\sum_{\substack{d|a\\ d>0}} d^s.$$

Each $L$-function is considered as a meromorphic function on $\boldsymbol{C}$ by the analytic continuation. A direct computation shows that $(*)$ may also be written:

$$a(T, [f])=(-1)^{k/2}\frac{(k-1)!}{(2k-2)!}(2\pi)^{k-1}\Delta(T)^{k-(3/2)}\frac{L(k-1, \chi)}{L_2(2k-2, f)} \cdot\sum_{\substack{v|\mathfrak{f}\\ v>0}} D(k-1, f, \varphi_{T,v}),$$

where

$$\varphi_{T,v}=(\mathfrak{f}v^{-1})^{2k-3}\prod_{p|\mathfrak{f}v^{-1},\, p:\text{ prime}}(1-\chi(p)p^{1-k})\vartheta_T^{(v)}.$$

In case $\Phi f \neq 0$ (i.e., $f=G_k$), (*) takes the following form:

$$a(T, [G_k])=(-1)^{k/2}\frac{(k-1)!}{(2k-2)!}2(2\pi)^{k-1}\Delta(T)^{k-(3/2)}\frac{L(k-1, \chi)}{\zeta(2k-2)}M(\mathfrak{f}).$$

This coincides with the formula of Maaß [6]. Theorem 1 for the case $\Phi f=0$ is also obtained by Böcherer [1] by a different method.

Exactly as in Part I [7], using the results of Shimura [11], Sturm [12], and Zagier [13] (see [7, Remark 2]) we obtain

**Theorem 2.** *Let $f$ be as in Theorem 1. Then for all $T \geqq 0$ we have:*

(1) *$a(T, [f])^\sigma = a(T, [f^\sigma])$ for all $\sigma \in \mathrm{Aut}(\boldsymbol{C})$.*

(2) *$a(T, [f])$ belong to $\boldsymbol{Q}(f)$.*

Prof. H. Maaß suggested that the author should study non-vanishing properties of $a(T, [f])$ in December 1980 in connection with Part I [7]. For this problem we have the following

**Theorem 3.** *Let $f$ be as in Theorem 1. Suppose that $T \geqq 0$ (as a binary quadratic form) represents 1 over $\boldsymbol{Z}$. Then: $a(T, [f]) \neq 0$.*

**§ 2. Modules in quadratic fields.** We sketch here the main tool for the proof of Theorem 1 which is not contained in Part I [7]. For basic properties of modules in quadratic fields (and the correspondence of them with binary quadratic forms), we refer to Borevich-Shafarevich [2]. Let $T$, $\Delta(T)$, and $\mathfrak{f}$ be as above. We put $K=\boldsymbol{Q}(\sqrt{-|2T|})$ and $K^\times = K-\{0\}$. For $0<c \in \boldsymbol{Z}$, let $\mathcal{O}_K(c)$ be the order of discriminant $-\Delta(T)c^2$ in $K$, and $\mathcal{M}_K(c)$ be the set of all full modules in $K$ with coefficient ring $\mathcal{O}_K(c)$. For each order $\mathcal{O}$ in $K$, put

$$\mathcal{M}_K(c, \mathcal{O})=\{M \in \mathcal{M}_K(c) \mid M \subset \mathcal{O}\}.$$

For each $m \geqq 1$ we put

$$\mathcal{M}_K(c, \mathcal{O}; m)=\{M \in \mathcal{M}_K(c, \mathcal{O}) \mid \mathrm{N}(M)=m\}$$

($\mathrm{N}(M)$ denoting the norm of $M$, i.e., $M\overline{M}=\mathrm{N}(M)\mathcal{O}_K(c)$). The finite abelian group $\mathcal{M}_K(c)/K^\times$ is denoted by $\mathcal{C}_K(c)$. For $c' \mid c$, let $\nu(c, c')$: $\mathcal{C}_K(c) \to \mathcal{C}_K(c')$ be the surjective homomorphism induced by $M \mapsto \mathcal{O}_K(c')M$. Let $H(-\Delta(T)c^2)$ be the group of the $\Gamma_1$-equivalence classes of primitive positive definite binary quadratic forms of discriminant $-\Delta(T)c^2$, which is isomorphic to $\mathcal{C}_K(c)$. Let $T_1, \cdots, T_h$ be representatives of all classes of $H(-\Delta(T)\mathfrak{f}^2)$. For each character $\psi$: $H(-\Delta(T)\mathfrak{f}^2) \to \boldsymbol{C}^\times$, we put $g_\psi = w^{-1}\sum_{1 \leqq j \leqq h}\psi(T_j)\vartheta_{T_j}$ ($w$ denoting the number of roots of unity in $\mathcal{O}_K(\mathfrak{f})$) and $g_\psi=\sum_{n \geqq 0} t(\psi, n)\mathrm{e}(nz)$. We consider $\psi$ also as a character of $\mathcal{C}_K(\mathfrak{f})$.

**Proposition.** *Let the notation be as above.*

(1) *$t(\psi, n)$ is multiplicative with respect to $n$; and*

$$t(\psi, n)=\sum_{M \in \mathcal{M}_K(\mathfrak{f}, \mathcal{O}_K(\mathfrak{f}); n)} \psi(M).$$

(2) *Let $p$ be a prime number and $p^\alpha$ (with $\alpha \geqq 0$) be the exact power of $p$ dividing $\mathfrak{f}$. Let $(p)=\mathfrak{p}\bar{\mathfrak{p}}$ (resp. $(p)=\mathfrak{p}^2$; $(p)=\mathfrak{p}$) be the prime decom-*

*position of $p$ in $\mathcal{O}_K(p^{-\alpha}\mathfrak{f})$ if* $\chi(p)=1$ (resp. $\chi(p)=0$; $\chi(p)=-1$); *the uniqueness of the prime decomposition of $p$ holds in $\mathcal{O}_K(p^{-\alpha}\mathfrak{f})$ since $p\nmid p^{-\alpha}\mathfrak{f}$. Then there exists a $P\in\mathcal{M}_K(\mathfrak{f},\mathcal{O}_K(p^{-\alpha}\mathfrak{f}))$ such that $P\mathcal{O}_K(p^{-\alpha}\mathfrak{f})=\mathfrak{p}$. Moreover $P$ can be taken so that $P=\bar{P}$ if $\chi(p)\neq 1$.*

(3) *Let $\{M\in\mathcal{M}_K(\mathfrak{f})\,|\,M\mathcal{O}_K(p^{-\alpha}\mathfrak{f})=\mathcal{O}_K(p^{-\alpha}\mathfrak{f})\}=\{M_j\,|\,j=1,\cdots,\kappa\}$, where $\kappa=p^{\alpha}(1-\chi(p)p^{-1})$ or $1$ according as $\alpha>0$ or $\alpha=0$. Then, for $2\alpha\leqq\delta\in\boldsymbol{Z}$, $\mathcal{M}_K(\mathfrak{f},\mathcal{O}_K(\mathfrak{f});p^{\delta})$ is equal to: $\{p^{\alpha}M_jP^{\delta-2\alpha-t}\bar{P}^t\,|\,1\leqq j\leqq\kappa,\ 0\leqq t\leqq\delta-2\alpha\}$ if $\chi(p)=1$, $\{p^{\alpha}M_jP^{\delta-2\alpha}\,|\,1\leqq j\leqq\kappa\}$ if $\chi(p)=0$, $\{p^{\alpha}M_jP^{(\delta/2)-\alpha}\,|\,1\leqq j\leqq\kappa\}$ if $\chi(p)=-1$ and $\delta$ even, $\phi$* (the empty set) *if $\chi(p)=-1$ and $\delta$ odd.*

(4) *Suppose $\delta\in\boldsymbol{Z}$, $0\leqq\delta\leqq 2\alpha$. Then $\mathcal{M}_K(\mathfrak{f},\mathcal{O}_K(\mathfrak{f});p^{\delta})=\{p^{\delta/2}M\,|\,M\in\mathcal{M}_K(\mathfrak{f})$ such that $M\mathcal{O}_K(\mathfrak{f}p^{-\delta/2})=\mathcal{O}_K(\mathfrak{f}p^{-\delta/2})\}$ for $\delta$ even; $\mathcal{M}_K(\mathfrak{f},\mathcal{O}_K(\mathfrak{f});p^{\delta})=\phi$ for $\delta$ odd. Moreover, for $0\leqq\delta\leqq 2\alpha$ and $\delta$ even, we have a bijection*

$$\mathcal{M}_K(\mathfrak{f},\mathcal{O}_K(\mathfrak{f});p^{\delta})\leftrightarrow(\mathcal{O}_K(\mathfrak{f}p^{-\delta/2})^{\times}/\mathcal{O}_K(\mathfrak{f})^{\times})\times(\mathrm{Ker}\,(\nu(\mathfrak{f},\mathfrak{f}p^{-\delta/2}))$$

*via $p^{\delta/2}\zeta M\leftrightarrow(\zeta\mathcal{O}_K(\mathfrak{f})^{\times},[M])$.* (Here $\zeta\in\mathcal{O}_K(\mathfrak{f}p^{-\delta/2})^{\times}$ and $[M]$ is the class of $C_K(\mathfrak{f})$ containing $M$. For a ring $R$ with 1, we denote by $R^{\times}$ the group of units.)

**Remark.** (i) Proposition determines all the values of $t(\psi,n)$ explicitly.

(ii) By Proposition, for example, we know that the "twisted" Epstein zeta function $L(s,\psi)=\sum_{n\geqq 1}t(\psi,n)n^{-s}$ has an Euler-product expression over all primes and each $p$-factor is a rational function of $p^{-s}$ with coefficients lying in $\boldsymbol{Q}(e^{2\pi\sqrt{-1}/h})$.

In case $\Phi f\neq 0$, i.e. $f=G_k$, by Proposition we have:

**Lemma.** *Put*

$$V(\psi)=\sum_{\substack{s|\mathfrak{f}\\ s>0}}M(\mathfrak{f}s^{-1})\sum_{\substack{t|s\\ t>0}}\mu(t)D(k-1,G_k,g_{\psi}^{(s/t)}).$$

*Then: $V(\psi)=0$ if $\psi$ is non-trivial, and*

$$V(\psi)=M(\mathfrak{f})\zeta(k-1)\zeta_K(0)\mathfrak{f}\prod_{p|\mathfrak{f}}(1-\chi(p)p^{-1})$$

*if $\psi$ is trivial. Here $\zeta_K(s)$ is the Dedekind zeta function of $K$.*

From this lemma the uniformity (for the both cases $\Phi f\neq 0$ and $\Phi f=0$) of the formula (*) follows. The proof in the case $\Phi f=0$ is based on the above Proposition and a multiplicative property of "sums of Kloosterman sums".

We note that $a(T,[f])$ have the unified expression for both cases $\Phi f\neq 0$ and $\Phi f=0$ for *all* $T$ by Theorem 1 above and Part I [7, Remark 2]. This suggests that the higher degree cases are in similar situations.

## References

[1] S. Böcherer: Über gewisse Siegelsche Modulformen zweiten Grades. Math. Ann., **261**, 23–41 (1982).

[2] Z. I. Borevich and I. R. Shafarevich: Number Theory. Academic Press (1966).

[3] N. Kurokawa: On Eisenstein series for Siegel modular groups. Proc. Japan Acad., **57**A, 51–55 (1981).
[4] ——: ditto. II. ibid., **57**A, 315–320 (1981).
[5] N. Kurokawa and S. Mizumoto: On Eisenstein series of degree two. ibid., **57**A, 134–139 (1981).
[6] H. Maaß: Die Fourierkoeffizienten der Eisensteinreihen zweiten Grades. Mat.-Fys. Medd. Danske Vid. Selsk., **34**, nr. 7 (1964).
[7] S. Mizumoto: Fourier coefficients of generalized Eisenstein series of degree two. I. Invent. math., **65**, 115–135 (1981).
[8] ——: ditto. II. Kodai Math. J. (to appear).
[9] ——: On Eisenstein series of degree two for Hilbert-Siegel modular groups. Proc. Japan Acad., **58**A, 33–36 (1982).
[10] ——: Integrality of certain algebraic values attached to modular forms. ibid., **59**A, 33–36 (1983).
[11] G. Shimura: The special values of the zeta functions associated with cusp forms. Comm. Pure Appl. Math., **29**, 783–804 (1976).
[12] J. Sturm: Special values of zeta functions and Eisenstein series of half integral weight. Amer. J. Math., **102**, 219–240 (1980).
[13] D. Zagier: Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields. Lect. Notes in Math., vol. 627, Springer-Verlag, pp. 105–169 (1977).