65. Wave Front Solution of Some Competition Models with Migration Effect

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1. Introduction. Consider the following semilinear hyperbolic system of partial differential equations

$$(1)\quad \begin{cases} \dfrac{\partial u}{\partial t}+\mu\dfrac{\partial u}{\partial x}=\left(a_1-b_1u-\dfrac{c_1v}{1+e_1u}\right)u\equiv f(u,v)u \\ \dfrac{\partial v}{\partial t}+\nu\dfrac{\partial v}{\partial x}=\left(a_2-\dfrac{b_2u}{1+e_2v}-c_2v\right)v\equiv g(u,v)v, \\ \qquad\qquad (x,t)\in(-\infty,+\infty)\times(0,+\infty), \end{cases}$$

where all the coefficients in (1) are real constants such that $\mu<\nu$, $a_i>0$, $b_i>0$, $c_i>0$ and $e_i\geqq 0$ $(i=1,2)$. Here, u and v are population densities of two species with distinct migration speed μ and ν respectively. Therefore, we consider nonnegative solutions only. In case of $e_1=e_2=0$, the system (1) becomes the classic Volterra-Lotka competition models. Yamaguti [4] considered the system (1) when f and g are linear functions of u and v, and he obtained the exact solutions by using Hirota's method [2]. By the computer simulations, we have found that the solutions given in [4] include the wave front solutions of the form

$$u(x,t)=u(z)=\left\{\gamma\frac{a_1}{b_1}\exp z\right\}\Big/\{1+\gamma\exp z\},$$

$$v(x,t)=v(z)=\frac{a_2}{c_2}\Big/\{1+\gamma\exp z\},\qquad z=Qx-\omega t,$$

where γ, Q and ω are some constants.

In this paper we shall show that the system (1) has unique (except modulo translation) wave front solutions joining two distinct states $P_1=(a_1/b_1,0)$ and $P_2=(0,a_2/c_2)$, where only one of two species exists.

2. Formulation of problem. In order to seek wave front solutions joining P_2 and P_1, put $(u(x,t),v(x,t))=(u(z),v(z))$ with $z=x-\sigma t$ $(\sigma\neq\mu,\nu)$. Then our problem is reduced to find solutions of ordinary differential equations of the form

$$(2)\quad \frac{du}{dz}=\frac{1}{\mu-\sigma}f(u,v)u\equiv f_1(u,v),\qquad \frac{dv}{dz}=\frac{1}{\nu-\sigma}g(u,v)v\equiv g_1(u,v)$$

with the boundary conditions

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(3) $(u(-\infty), v(-\infty))=(0, a_2/c_2) \qquad (u(+\infty), v(+\infty))=(a_1/b_1, 0)$.

Now, by solving $f(u, v)=0$ and $g(u, v)=0$ with respect to u and v respectively, we have two functions $p(u)$ and $q(v)$ of the form

(4)
$$v=p(u)=-\frac{b_1}{c_1}\left(u-\frac{a_1}{b_1}\right)(e_1u+1),$$
$$u=q(v)=-\frac{c_2}{b_2}\left(v-\frac{a_2}{c_2}\right)(e_2v+1).$$

For these functions $p(u)$ and $q(v)$, we assume that

(5)
$$\max\left\{p(u)\,|\,0\leqq u\leqq \frac{a_1}{b_1}\right\}<\frac{a_2}{c_2},$$
$$\max\left\{q(v)\,|\,0\leqq v\leqq \frac{a_2}{c_2}\right\}=d<\frac{a_1}{b_1},$$

and define the domain $\mathscr{D}$ by

(6) $$\mathscr{D}=\left\{(u, v)\,|\,p(u)<v<\frac{a_2}{c_2}\right\}\cap\left\{(u, v)\,|\,q(v)<u<\frac{a_1}{b_1}\right\}.$$

Hence by (4) we have

(7) $$f(u, v)<0, \quad g(u, v)<0, \quad (u, v)\in\mathscr{D}.$$

It can be easily seen from the phase plane analysis that there exist solutions of (2) and (3) only when the traveling speed σ satisfies the conditions

(8) $$\mu<\sigma<\nu.$$

In view of (5) and (8), we know that the dynamical system (2) has at least four critical points among which $P_0=(0, 0)$, P_1 and P_2 are saddle.

3. Existence of wave front solutions. We shall show the existence of wave front solutions by the following lemmas.

Lemma 1. *For any fixed σ satisfying* (8) *there exists unique trajectory $v=v(u, \sigma)$, in $\mathscr{D}$ through the point P_2, of equation*

(9) $$\frac{dv}{du}=h(\sigma)\frac{g(u, v)v}{f(u, v)u}, \qquad \textit{where } h(\sigma)=\frac{\mu-\sigma}{\nu-\sigma},$$

and it satisfies the inequality

(10) $$v(u, \sigma_2)<v(u, \sigma_1)$$

for $\mu<\sigma_1<\sigma_2<\nu$ on the interval on u where $v(u, \sigma_1)$ and $v(u, \sigma_2)$ are defined.

Proof. For any point $P=(u, v)$, we put

$$J(P)=\begin{pmatrix}(\partial f_1/\partial u)(u, v) & (\partial f_1/\partial v)(u, v)\\ (\partial g_1/\partial u)(u, v) & (\partial g_1/\partial v)(u, v)\end{pmatrix},$$

where f_1 and g_1 are defined in (2). Then for $P_2=(0, a_2/c_2)$ we have

$$J(P_2)=\begin{pmatrix}\{c_1/(\mu-\sigma)\}\{(a_1/c_1)-(a_2/c_2)\} & 0\\ \{1/(\sigma-\nu)\}\{(a_2b_2)/(a_2e_2+c_2)\} & a_2/(\sigma-\nu)\end{pmatrix}.$$

Denote the eigenvector of $J(P_2)$ corresponding to the positive eigenvalue $\{c_1/(\mu-\sigma)\}\{(a_1/c_1)-(a_2/c_2)\}$ by ${}^t(1, m(\sigma))$, where $m(\sigma) =\{(a_2b_2c_2)/(a_2e_2+c_2)\}\{h(\sigma)/((a_2c_1-a_1c_2)-a_2c_2h(\sigma))\}$. By (6) and (8) we have $\{(-b_2)/(a_2e_2+c_2)\}<m(\sigma)<0$, $\lim_{\sigma\to\mu+0} m(\sigma)=0$ and $\lim_{\sigma\to\nu-0} m(\sigma)$

$=(-b_2)/(a_2e_2+c_2)$. For any fixed σ satisfying (8), it is well known fact that the trajectory of (9), leaving the saddle point P_2 in the direction of eigen vector ${}^t(1, m(\sigma))$, is unique (see [3]). If $\sigma_1<\sigma_2$ then $m(\sigma_2)<m(\sigma_1)$ holds, and for sufficiently small $\delta>0$, we have $v(u,\sigma_2)<v(u,\sigma_1)$ on the interval $0<u<\delta$. We now suppose that the inequality (10) does not hold. Then, for some $u_0>0$, we have $v(u_0,\sigma_2)=v(u_0,\sigma_1)$ and $v(u,\sigma_2)<v(u,\sigma_1)$ $(0<u<u_0)$. By putting $v(u_0,\sigma_2)=v(u_0,\sigma_1)=v_0$ and by (7) and (9), we have $(dv/du)(u_0,\sigma_2)-(dv/du)(u_0,\sigma_1)<0$ for $(u_0,v_0)\in\mathscr{D}$ since $h(\sigma_2)<h(\sigma_1)$. On the other hand, we have $\{(v(u,\sigma_1)-v(u_0,\sigma_1))/(u-u_0)\}<\{(v(u,\sigma_2)-v(u_0,\sigma_2))/(u-u_0)\}$ for $0<u<u_0$. Hence we obtain $(dv/du)(u_0,\sigma_1)\leqq(dv/du)(u_0,\sigma_2)$ as u tends to u_0, which contradicts the fact that $(dv/du)(u_0,\sigma_2)-(dv/du)(u_0,\sigma_2)<0$. Q. E. D.

Lemma 2. *The trajectory $v(u,\sigma)$ of* (9) *in $\mathscr{D}$ through P_2 is continuous with respect to σ, that is, for any σ_0 satisfying* (8) *we obtain*

$$\lim_{\sigma\to\sigma_0} v(u,\sigma)=v(u,\sigma_0). \tag{11}$$

Proof. See Proposition 4.5 in Aronson and Weinberger [1].

Lemma 3. *The trajectory $v=v(u,\sigma)$ in $\mathscr{D}$ through P_2 passes through some point $(a_1/b_1, v_0)$ with $0<v_0<(a_2/c_2)$ when σ is sufficiently close to μ.*

Proof. Put $v=k(u)=-Cu+(a_2/c_2)$. Here, we choose positive constant C small so that $\{(u,k(u))\,|\,0<u<(a_1/b_1)\}\in\mathscr{D}$ holds. Then, by (5) and (7), there exists some positive constant M such that $0<\{(g(u,v)v)/(f(u,v)u)\}\leqq M$ holds on $L_1=\{(u,v)\,|\,v=k(u),\ 0<u<(a_1/b_1)\}$. Then, if σ is sufficiently close to μ we see from (9) that $-C<dv/du=h(\sigma)(g(u,v)v/f(u,v)u)<0$ holds on L_1. On the other hand, $dv/du=h(\sigma)(g(u,v)v/f(u,v)u)<0$ on $L_2=\{(u,v)\,|\,0<u\leqq a_1/b_1,\ v=a_2/b_2\}$. Hence we can easily derive the conclusion of this lemma. Q. E. D.

Lemma 4. *Put $\mathcal{E}=\{\sigma\in(\mu,\nu)\,|\,0\leqq v(a_1/b_1,\sigma)<a_2/c_2,\ (u,v(u,\sigma))\in\mathscr{D}$ for $0<u<(a_1/b_1)\}$ and define σ^* by $\sigma^*=\sup\{\sigma\,|\,\sigma\in\mathcal{E}\}$, then $\sigma^*\in\mathcal{E}$ and $\lim_{u\to(a_1/b_1)-0} v(u,\sigma^*)=0$ hold.*

Proof. Since $\mathcal{E}\neq\phi$, by Lemma 3, we have $\mu<\sigma^*$. Suppose that $\sigma^*=\nu$. In view of (5), we choose two numbers u_1 and u_2 such that $d<u_1<u_2<(a_1/b_1)$ and put $\mathscr{A}=\{(u,v)\,|\,u_1<u<u_2\}\cap\mathscr{D}$, then for some positive constant α we have $\alpha\leqq\{(g(u,v)v)/(f(u,v)u)\}$, where $(u,v)\in\mathscr{A}$. Hence, if σ_0 is sufficiently close to ν we have $dv/du=h(\sigma_0)(g(u,v)v/f(u,v)u)\leqq\alpha h(\sigma_0)<-(2a_2)/\{(u_2-u_1)c_2\}$. Therefore the trajectory $v=v(u,\sigma_0)$ in $\mathscr{D}$ through P_2 cannot cross the straight line $u=u_2$, which contradicts the fact that $\sigma_0\in E$. Hence we have $\sigma^*<\nu$. Next, suppose that $\sigma^*\notin\mathcal{E}$. Then the trajectory $v=v(u,\sigma^*)$ in $\mathscr{D}$ through P_2 intersects the curve $v=p(u)$ at some point $(\eta_1, p(\eta_1))$. For a point $(\eta_2, p(\eta_2))$ with $\eta_1<\eta_2<(a_1/b_1)$, the trajectory $v=v(u,\sigma^*)$ in $\mathscr{D}$ through $(\eta_2, p(\eta_2))$ must intersect the segment L_2 which is defined in the proof of Lemma 3.

By the continuous dependency of solution of initial value problem of ordinary differential equation, the trajectory $v=v_1(u, \sigma_1)$ in $\mathcal{D}$ through $(\eta_2, p(\eta_2))$ must intersect L_2 when σ_1 $(\in \mathcal{E})$ is sufficiently close to σ^*. Hence, the trajectory $v=v_1(u, \sigma_1)$ in $\mathcal{D}$ through P_2 intersects the above trajectory $v=v_1(u, \sigma_1)$ at some point in $\mathcal{D}$. This, however, contradicts the uniqueness of trajectory through the ordinary point. Therefore we have $\sigma^* \in \mathcal{E}$. Next, suppose that $\lim_{u\to(a_1/b_1)-0} v(u, \sigma^*)=v^*>0$. For sufficiently small $\varepsilon>0$, by Lemmas 1 and 2, there exist v_2 and σ_2 such that $\sigma^*<\sigma_2<\nu$, $v^*-\varepsilon<v_2<v^*$ and $\lim_{u\to(a_1/b_1)-0} v(u, \sigma_2)=v_2$ are satisfied. Hence, we have $v(a_1/b_1, \sigma_2)=v_2>0$ since $v(u, \sigma_2)$ is continuous on the interval $(0, a_1/b_1]$ with respect to u. This, however, contradicts the definition of σ^*. Q. E. D.

Lemma 4 yields the following theorem.

Theorem. *If the condition* (5) *is satisfied, then, only for* σ^* *given in Lemma* 4, *there exist unique* (except modulo translation) *solutions* $(u(x, t), v(x, t))=(u(z), v(z))$, *where* $z=x-\sigma^* t$, *of the following equations*

$$\frac{du}{dz}=\frac{1}{\mu-\sigma^*}f(u, v)u, \qquad \frac{dv}{dz}=\frac{1}{\nu-\sigma^*}g(u, v)v$$

with the boundary condition (3).

Remark. If the condition (5) is violated, it is easily seen from the phase plane analysis that there exists no trajectory of (2) joining P_2 and P_1.

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