62. On the Resolution Process of Normal Gorenstein Surface Singularity with $p_a \leqq 1$

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Let (V, p) be a normal Gorenstein surface singularity over $\boldsymbol{C}$ and ψ ; $(\tilde{V}, A) \to (V, p)$ be a resolution of (V, p) with the exceptional set A. The arithemetic genus of (V, p) is the integer $p_a(V, p)$ defined by $p_a(V, p) = \sup\{p_a(D) \mid \text{divisor } D \text{ on } \tilde{V},\ D > 0,\ |D| \subset A\}$. The geometric genus of (V, p) is the integer $p_g(V, p)$ defined by $p_g(V, p) = \dim_C R^1\psi_* O_{\tilde{V}}$ (see [6]).

The singularity (V, p) is called *rational* (resp. *elliptic*) if $p_a(V, p) = 0$ (resp. $p_a(V, p) = 1$). The *Zariski's canonical resolution*, which we simply call Z. C. R. here, of (V, p) is the resolution obtained by the composition of blowing-up with center maximal ideal followed by normalization (see e.g., [3]). The singularity (V, p) is called *absolutely isolated* if all the normalizations in Z. C. R. of (V, p) are trivial. The purpose of this note is to announce the recent results on the Z. C. R. for the normal Gorenstein surface singularity with $p_a \leqq 1$. The details will be published elsewhere.

§1. Absolute isolatedness of elliptic singularity. Theorem 1. *Let (V, p) be a normal Gorenstein surface singularity with $p_a \leqq 1$. Then Z. C. R. is obtained by the composition of blowing-ups as follows:*

$$(*)\qquad \begin{array}{ccccccccc} U = U_0 & \longleftarrow & U_1 & \longleftarrow & U_2 & \longleftarrow & \cdots & \longleftarrow & U_N \\ \cup & \psi_1 & \cup & \psi_2 & \cup & \psi_3 & & \psi_N & \cup \\ V = V_0 & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow & \cdots & \longleftarrow & V_N \end{array}$$

where $V \subset U$ is the minimal embedding, ψ_i the blowing-up of U_{i-1} with smooth center $C_i \subset V_{i-1}$, and V_i the strict transformation of V_{i-1}, $1 \leqq i \leqq N$. Moreover we have: There is an integer M $(\leqq N)$ such that (i) *V_k is normal for $k \leqq M$.* (ii) *ψ_k is a blowing-up with point center p_k, such that (V_{k-1}, p_k) is Gorenstein of maximal embedding dimension* [4] *of multiplicity $\geqq 3$ for $k \leqq M$.* (iii) *At each stage, in which V_i is normal, there is at most one non-rational singularity.* (iv) $\operatorname{mult}_q V_M \leqq 2$ *for any point $q \in V_M$.* (v) [5] *In the Z. C. R. for the singularity of V_M, the normalizations are obtained by one blowing-up along* (reduced) $\boldsymbol{P}^1$.

There are $p_g(V, p) - M$ blowing-ups along $\boldsymbol{P}^1$ in the diagram $(*)$. On the other hand, we have

Theorem 2 [1]. *Let (V, p) be a normal surface singularity of multiplicity two. Then (V, p) is absolutely isolated if and only if (V, p)*

is rational.

Hence, the normal surface Gorenstein elliptic singularity (V, p) is absolutely isolated if and only if the singularities of V_M are rational in the diagram $(*)$. The following theorem extends the results [2], [8] about the absolute isolatedness in the case of $p_g \leqq 2$.

Theorem 3. *Let (V, p) be a normal Gorenstein surface elliptic singularity. Let $(\tilde{V}, A) \to (V, p)$ be the minimal resolution of (V, p) and E the minimal elliptic cycle* [2]. *Then the singularity (V, p) is absolutely isolated if and only if $E^2 \leqq -3$.*

§2. The singularities satisfying $L(V, p) = p_g(V, p)$. We shall consider the problem "*What is the condition in the resolution $(*)$ for (V, p) to satisfy $p_a \leqq 1$?*" A sufficient condition is given by Theorems 4 and 9 below.

Theorem 4 (Yau [7], see [5] for this formulation). *The normal Gorenstein surface singularity (V, p) satisfies the inequality $p_a \leqq 1$ if the equality $L(V, p) = p_g(V, p)$ holds. Here, the integer $L(V, p)$ is defined by $L(V, p) = \min\{\alpha \in \boldsymbol{Z} \mid -K_{\tilde{V}} \leqq \alpha Y_\psi\}$, where Y_ψ is the maximal ideal cycle for ψ and $K_{\tilde{V}}$ the canonical divisor on $\tilde{V}$ whose supports are in A* [5].

In the case of multiplicity two, the existence of the resolution $(*)$ gives a characterization of the singularity with $p_a \leqq 1$ as follows.

Theorem 5 [5]. *Let (V, p) be a normal surface singularity of multiplicity two. The following three conditions are equivalent.* (i) $p_a \leqq 1$. (ii) *Z.C.R. is obtained as in* $(*)$ with $M = 0$. (iii) $L(V, p) = p_g(V, p)$.

Example 6. $(\{z^3 = x^4 + y^6\}, o)$ has the invariants $p_g = 3$, $L = p_a = 2$, and has a resolution satisfying $(*)$.

Introducing the following notion, we shall cosider the gap between the condition $(*)$ and the equality $L(V, p) = p_g(V, p)$.

Definition 7 (*Starting points*). In the diagram $(*)$, the normal point q is called *starting point* if q is one of the following points. (i) p_k, $k \leqq M$. (ii) the non-rational point in V_M. (iii) the non-rational points which appear after the blowing-up along $\boldsymbol{P}^1$.

Remark 8. There are $p_g(V, p)$ starting points in the diagram $(*)$.

Theorem 9. *Let (V, p) be a normal Gorenstein surface singularity. Then the following conditions are equivalent.* (i) $L(V, p) = p_g(V, p)$. (ii) *Z.C.R. is obtained by $(*)$ and there is a smooth curve l in U, such that the strict transforms of l in U_i contain all the starting points.*

By elementary computations, we obtain

Theorem 10. *Let (V, p) be a normal Gorenstein surface singularity with $p_a \leqq 1$ of* $\mathrm{mult}_p V \leqq 4$. *Then the equality $L(V, p) = p_g(V, p)$ holds.*

Note that the Gorenstein local ring of maximal embedding dimension is a complete intersection if and only if $\mathrm{mult}_p V \leqq 4$.

Hence, together with Theorem 4, we have a geometric characterization (in the sense of (ii) in Theorem 9) of a complete intersection singularity with $p_a \leqq 1$.

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