7. *On a Question Posed by Huckaba-Papick*

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1. Introduction. Let R be a commutative integral domain with identity, and let x be an indeterminate. By $c(f)$ we denote the ideal of R generated by the coefficients of f for an element f of $R[x]$ (the 'content' of f). Let K be the quotient field of R. We denote the R-submodule $\{b \in K; b\mathfrak{A} \subset R\}$ of K for an ideal $\mathfrak{A}$ of R by $\mathfrak{A}^{-1}$. We set $S=\{f \in R[x]; c(f)=R\}$ and $U=\{f \in R[x]; c(f)^{-1}=R\}$. These are multiplicative systems of $R[x]$. Hence we can define subrings $R[x]_S$ and $R[x]_U$ of $K(x)$; $R[x]_S \subset R[x]_U$. If we have $\mathfrak{A}\mathfrak{A}^{-1}=R$ for each finitely generated ideal $\mathfrak{A}$ of R, R is said to be a *Prüfer ring*. If each finitely generated ideal of R is a principal ideal, R is said to be a *Bezout ring*. If R is a Bezout ring, then it is a Prüfer ring. Huckaba-Papick studied in [2], the following problems: When does $R[x]_S=R[x]_U$ hold? and when is $R[x]_U$ a Prüfer ring? And they posed the open question:

Question ([2, Remark (3.4), (c)]). If $R[x]_U$ is a Prüfer ring, is it a Bezout ring?

The main purpose of this paper is to give an affirmative answer to this question. We prove the following result:

Theorem 1. *If $R[x]_U$ is a Prüfer ring, it is a Bezout ring.*

Among other things, Huckaba-Papick prove the following result in [2, Theorem (3.1), (c)]: If R is a Krull ring, then $R[x]_U$ is a Bezout ring. But their proof does not seem to be complete. So we prove the following result for the sake of completeness.

Theorem 2. *If R is a Krull ring, then $R[x]_U$ is a principal ideal ring. Conversely, if $R[x]_U$ is a Krull ring, then R is also a Krull ring.*

2. Proofs of Theorems 1 and 2. We denote the ideal $\{r \in R; rb \in (a)\}$ of R for two elements a, b of R by $(a:b)$ as in [2]. Let $\mathcal{P}(R)$ be the set of prime ideals of R which are minimal prime ideals over $(a:b)$ for some elements a, b of R.

Lemma 3 ([3, Theorem E]).

(1) $U=R[x]-\bigcup_{P \in \mathcal{P}(R)} PR[x]$;

(2) $R=\bigcap_{P \in \mathcal{P}(R)} R_P$.

Lemma 4 ([1, § 18, Exercise 12]). *Let V be a valuation ring of $K(x)$ of the form $R[x]_Q$ for a prime ideal Q of $R[x]$. Then we have either* (1) *or* (2) *of the following*:

(1) *There exists an irreducible polynomial f of $K[x]$ such that* $V=K[x]_{fK[x]}$;

(2) *there exists a prime ideal P of R such that R_P is a valuation ring of K and such that* $V=R[x]_{PR[x]}$.

Lemma 5 ([1, (34.9) Corollary]). *Let R be an integrally closed ring* (in K). *Then we have*

$$fK[x]\cap R[x]=fc(f)^{-1}R[x]$$

for a nonzero element f of $R[x]$.

Lemma 6. *Let $\mathfrak{A}$ be a finitely generated ideal of R. Then we have $\mathfrak{A}^{-1}R[x]_U=(\mathfrak{A}\,R[x]_U)^{-1}$ as $R[x]_U$-ideals.*

Proof. There exist a finite number of elements $a_1, \cdots, a_n$ of R such that $\mathfrak{A}=(a_1, \cdots, a_n)$. Let $\xi\in(\mathfrak{A}R[x]_U)^{-1}$. There exist elements $u\in U$ and $f_i\in R[x]$ for $1\leq i\leq n$ such that $\xi ua_i=f_i$. It follows $\xi u\in\mathfrak{A}^{-1}R[x]$, hence $(\mathfrak{A}R[x]_U)^{-1}\subset\mathfrak{A}^{-1}R[x]_U$. The opposite inclusion relation $(\mathfrak{A}R[x]_U)^{-1}\supset\mathfrak{A}^{-1}R[x]_U$ is immediate.

Lemma 7 ([1, a part of (19.15) Theorem]). *Let R be an integrally closed ring, and let P be a prime ideal of R. Then the following* (1) *and* (2) *are equivalent:*

(1) *R_P is a valuation ring of K.*

(2) *Each prime ideal of $R[x]$ contained in $PR[x]$ is the extension of a prime ideal of R.*

Lemma 8 ([2, Lemma 3.0]). *Let R be an integrally closed ring, and let W be a multiplicative system of $R[x]$. If each prime ideal of $R[x]_W$ is the extension of a prime ideal of R, then $R[x]_W$ is a Bezout ring.*

Proof of Theorem 1. Let P be an element of $\mathscr{P}(R)$. We have $PR[x]\cap U=\phi$ by Lemma 3(1). Since $R[x]_U$ is a Prüfer ring, $(R[x]_U)_{PR[x]_U}$ is a valuation ring of $K(x)$. Since $(R[x]_U)_{PR[x]_U}\cap K=R_P$, R_P is a valuation ring of K. Hence R is an integrally closed ring by Lemma 3(2). Let $\mathfrak{p}$ be a proper prime ideal of $R[x]_U$: $R[x]_U\supsetneqq\mathfrak{p}\supsetneqq(0)$. There exists a prime ideal Q of $R[x]$ such that $Q\cap U=\phi$ and such that $QR[x]_U=\mathfrak{p}$. If $Q\cap R=P$ is not a zero ideal of R, we have $(R[x]_U)_{\mathfrak{p}}=R[x]_{PR[x]}$ by Lemma 4, and hence $\mathfrak{p}=PR[x]_U$. Therefore $\mathfrak{p}$ is the extension of a prime ideal of R. Next we suppose that $Q\cap R=(0)$, and we will derive a contradiction. We have $(R[x]_U)_{\mathfrak{p}}=K[x]_{fK[x]}$ for an irreducible polynomial f of $K[x]$ by Lemma 4. We can take f in $R[x]$. We have $Q=fc(f)^{-1}R[x]$ by Lemma 5. We have $\mathfrak{p}=f(c(f)R[x]_U)^{-1}$ by Lemma 6. Since $c(f)R[x]_U$ is a finitely generated ideal, hence an invertible ideal of $R[x]_U$, we have $1=\sum_{i=1}^{n}F_iG_i$ for $F_i\in c(f)R[x]_U$ and for $G_i\in(c(f)R[x]_U)^{-1}$ $(1\leq i\leq n)$. It follows that $(c(f)R[x]_U)^{-1}=(G_1, \cdots, G_n)\times R[x]_U$. Therefore $\mathfrak{p}$ is a finitely generated ideal of $R[x]_U$. We can find elements $f_1, \cdots, f_n$ of $R[x]$ such that $\mathfrak{p}=(f_1, \cdots, f_n)R[x]_U$. We set $h=f_1+f_2x^{1+d_1}+f_3x^{2+d_1+d_2}+\cdots+f_nx^{(n-1)+d_1+\cdots+d_{n-1}}$, where d_i is the degree of f_i. Since h is an element in $\mathfrak{p}\cap R[x]=Q$, it is also in $PR[x]$ for some $P\in\mathcal{P}(R)$ by Lemma 3(1). Since $c(f_i)\subset c(h)\subset P$, we have $f_i\in PR[x]_U$ for $1\leq i\leq n$. It follows that $\mathfrak{p}\subset PR[x]_U$, and hence $Q\subset PR[x]$. We have $Q=(Q\cap R)R[x]=(0)$ by Lemma 7: a contradiction. Therefore each prime ideal of $R[x]_U$ is the extension of a prime ideal of R. Accordingly $R[x]_U$ is a Bezout ring by Lemma 8. This completes the proof of Theorem 1.

We prepare one more lemma for the proof of Theorem 2.

Lemma 9. *We have* $R[x]_U\cap K=R$.

Proof. Let b be a nonzero element of K contained in $R[x]_U$. We have $b=f/u$ with $f\in R[x]$ and $u\in U$. Since $c(u)^{-1}=R$ and $bu=f\in R[x]$, we have $b\in R$.

Proof of Theorem 2. Let R be a Krull ring, and let $\{P_\lambda;\lambda\in\Lambda\}$ be the set prime ideals of R of height 1. We have $\mathcal{P}(R)=\{P_\lambda;\lambda\in\Lambda\}\cup\{(0)\}$ as in the proof of [2, Theorem (3.1)(c)]. Let Q be a nonzero prime ideal of $R[x]$ such that $Q\cap U=\phi$. We take a nonzero element h of Q. There are only finitely many $P_i\in\mathcal{P}(R)$ such that $h\in P_iR[x]$ $(1\leq i\leq t)$. If $Q\not\subset\bigcup_{i=1}^t P_iR[x]$, we choose $k\in Q-\bigcup_{i=1}^t P_iR[x]$. We set $s=\deg h$. Since $h+kx^{s+1}\in Q$, we have $h+kx^{s+1}\in P'R[x]$ for some $P'\in\mathcal{P}(R)$ by Lemma 3(1). Since $h\in P'R[x]$, we have $P'=P_j$ for some j. It follows that $k\in P_jR[x]$: a contradiction. Therefore we have $Q\subset\bigcup_{i=1}^t P_iR[x]$, and hence $Q\subset P_jR[x]$ for some j. Since R_{P_j} is a valuation ring of K, we have $Q=(Q\cap R)R[x]=P_jR[x]$ by Lemma 7. Hence each nonzero prime ideal of $R[x]_U$ is the extension of some $P_j\in\mathcal{P}(R)$. It follows that $R[x]_U$ is a Bezout ring by Lemma 8 and that $\dim R[x]_U\leq 1$. Since $R[x]$, hence also $R[x]_U$ are Krull rings, $R[X]_U$ is a principal ideal ring. Conversely, let $R[x]_U$ be a Krull ring. Then we see that R is a Krull ring by Lemma 9. We have completed the proof of Theorem 2.

Finally, a similar argument to the proof of Theorem 2 shows the following.

Proposition 10. If R is a generalized Krull ring, then $R[x]_U$ is a Bezout ring which is a generalized Krull ring of $\dim\leq 1$. Conversely, if $R[x]_U$ is a generalized Krull ring, then R is also a generalized Krull ring.

References

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