60. *Lie Groups and Lie Algebras with generalized Bose-Fermi Symmetric Parameters*

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1. Introduction. Lie superalgebras and Lie supergroups have been utilized extensively in physics. In those theories, Grassmann algebras play an important role. In fact, the theory of supermanifolds and supergroups are formulated on the basis of Grassmann algebras (Rogers [5], [6]). They have been also used in describing Fermi systems (Berezin [1]). Ohnuki and Kamefuchi [3] have considered generalized Grassmann numbers and generalized Bose numbers to describe para-Fermi and para-Bose systems (see also [4]). Scheunert [7] introduced algebras satisfying certain conditions which are considered to be the most general with respect to the concept of commutativity. We call them the σ-commutative algebras.

In this paper, we develop the theory of matrices whose entries are elements of a σ-commutative algebra and study the groups and the algebras consisting of those matrices. The details of the results will appear elsewhere.

2. σ-symmetry. In this section, we give some basic concepts which were basically formulated by Scheunert [7].

Let k be a field and let G be an abelian group. A mapping $\sigma: G \times G \to k$ is called a *sign* (commutation factor in terms of [7]) of G, if it satisfies

(1) $\sigma(\alpha+\beta, \gamma)=\sigma(\alpha, \gamma)\sigma(\beta, \gamma)$,

(2) $\sigma(\alpha, \beta)\sigma(\beta, \alpha)=1$,

for any $\alpha, \beta, \gamma \in G$. The pair (G, σ) is called a *signed group*.

Throughout this paper, a field k and a signed group (G, σ) are fixed. As is easily seen, $\sigma(\alpha, \alpha)$ is either 1 or -1 for any $\alpha \in G$. An element α of G such that $\sigma(\alpha, \alpha)=1$ (resp. -1) is called *even* (resp. *odd*).

A G-graded (associative) algebra $A=\oplus_{\alpha\in G} A_\alpha$ is called *σ-commutative* or *σ-symmetric* if $ab=\sigma(\alpha, \beta)ba$ for any $\alpha, \beta \in G$, $a \in A_\alpha$, $b \in A_\beta$.

Let $V=\oplus_{\alpha\in G} V$ be a G-graded vector space over k. Let $T(V)$ be the tensor algebra of V over k and I be the ideal of $T(V)$ generated by the elements of the form $x\otimes y-\sigma(\alpha, \beta)y\otimes x$ with $x \in V_\alpha$ and $y \in V_\beta$. The

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quotient algebra $\tilde{U}(V)=T(V)/I$ is σ-commutative and is called the *σ-symmetric algebra* of V over k.

Let A and B be G-graded algebras over k. The *graded tensor product* $A\otimes_k B$ of A and B is a G-graded algebra $\oplus_{\alpha\in G}(\oplus_{\beta+\gamma=\alpha}A_\beta\otimes_k B_\gamma)$ in which multiplication is defined by

$$(a\otimes b)(c\otimes d)=\sigma(\beta,\gamma)(ac\otimes bd)$$

for $\beta,\gamma\in G$ and $a\in A$, $b\in B_\beta$, $c\in A_\gamma$, $d\in B$. If A and B are σ-commutative, so is $A\otimes_k B$.

3. Super determinant. In this section, $F=\oplus_{\alpha\in G}F_\alpha$ is a G-graded σ-commutative algebra over k with identity element 1. A finite set I is called a *G-set* if it is linearly ordered and a grade $g(i)\in G$ is given for every $i\in I$. $|I|$ denotes the cardinality of I. I is called *even* (resp. *odd*) if the grade of every element of I is even (resp. odd). For a G-set I, we define another G-set $-I=\{-i\,|\,i\in I\}$ in such a way that $-i<-j$ if $i>j$ and $g(-i)=-g(i)$. We also define $g(I)=\sum_{i\in I}g(i)$.

Definition 1. Let I and J be G-sets. An *$I\times J$-matrix over F* is a $|I|\times|J|$-matrix $M=(M^i_j)$ with $M^i_j\in F_{g(i)-g(j)}$ for $i\in I$ and $j\in J$.

Let I, J and K be G-sets and let M and N be an $I\times J$-matrix and a $J\times K$-matrix over F respectively. Then the product MN is an $I\times K$-matrix over F. In particular, $I\times I$-matrices are closed under multiplication and form a Lie algebra over k by the operation $[M,N]=MN-NM$. It is called the *general linear Lie algebra with parameters in F* and is denoted by $gl(I,F)$.

Let I and J be G-sets such that $|I|=|J|$. To define the determinants of $I\times J$-matrices, first suppose that J is odd. Let x^j $(j\in J)$ be indeterminates of grade $g(j)$ and let $k[x]=k[x^j\,;\,j\in J]$ be the σ-symmetric algebra generated by x^j $(j\in J)$ over k.

Definition 2. The determinant $\det M$ of an $I\times J$-matrix M (J is odd) is defined by the equation in $F\otimes_k k[x]$;

$$\sum_\pi \prod_{i\in I} M^i_{\pi(i)}x^{\pi(i)}=\det M\cdot\prod_{j\in J}x^j,$$

where π ranges over all the bijections of I to J, and $\prod_{i\in I}$ and $\prod_{j\in J}$ mean the ordered product, for example, $\prod_{j\in J}x^j=x^{j_1}x^{j_2}\cdots x^{j_n}$ if $j_1<\cdots<j_n$.

Next, let J be an even G-set. We extend the signed group (G,σ) to $(\bar{G},\bar{\sigma})$ as follows: $\bar{G}=G\oplus Z_2$ and $\bar{\sigma}((\alpha,m),(\beta,n))=\sigma(\alpha,\beta)(-1)^{mn}$ for $\alpha,\beta\in G$ and $m,n\in Z_2$. Let y^j $(j\in J)$ be indeterminates of odd grade $(g(j),1)\in\bar{G}$ and let $k[y]=k[y^j\,;\,j\in J]$ be the $\bar{\sigma}$-symmetric algebra generated by y^j $(j\in J)$ over k.

Definition 3. The determinant $\det M$ of an $I\times J$-matrix M (J is even) is defined by the equation in $F\otimes_k k[y]$;

$$\sum_\pi \prod_{i\in I} M^i_{\pi(i)}y^{\pi(i)}=\det M\cdot\prod_{j\in J}y^j,$$

where π, $\prod_{i\in I}$ and $\prod_{j\in J}$ are the same as in Definition 2.

An $I \times J$-matrix M over F is called *square*, if $I = I_1 \cup I_2$, $J = J_1 \cup J_2$, I_1 and J_1 are even, I_2 and J_2 are odd and $|I_1| = |J_1|$, $|I_2| = |J_2|$. In this case, we write

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{$*$}$$

where A (resp. B, C, D) is an $I_1 \times J_1$ (resp. $I_1 \times J_2$, $I_2 \times J_1$, $I_2 \times J_2$)-matrix.

Definition 4. For a square matrix M over F of the form $(*)$ the *super determinant* sdet M of M is defined by

$$\operatorname{sdet} M = \begin{cases} 0, & \text{if } \det A \text{ or } \det D \text{ is not invertible,} \\ \sigma(g(J_1) - g(J_2),\ g(J_2) - g(I_2)) \cdot \det A \cdot (\det(D - CA^{-1}B))^{-1}, & \text{otherwise.} \end{cases}$$

Our definition is a generalization of those which are given by [4] and Leites [2].

Theorem 1. *Let M be a square $I \times J$-matrix over F. Then M is invertible if and only if* $\operatorname{sdet} M \neq 0$.

Theorem 2. *Let M be a square $I \times J$-matrix and N be a square $J \times K$-matrix. Then we have*

$$\operatorname{sdet} MN = \operatorname{sdet} M \cdot \operatorname{sdet} N.$$

Definition 5. For an $I \times J$-matrix M over F, a $(-J) \times (-I)$-matrix M^T called the *super transpose* of M is defined by

$$M^T{}^{-j}_{-i} = \sigma(g(i),\ g(j) - g(i)) M^i_j.$$

For an $I \times I$-matrix M the *super trace* str M of M is defined by

$$\operatorname{str} M = \textstyle\sum_{i \in I} \sigma(g(i), g(i)) M^i_i.$$

4. **Linear Lie groups and algebras with σ-symmetric parameters.** In this section, k is the real number field and F is a (σ-commutative) G-graded Banach algebra in the following sense.

Definition 6. A G-graded algebra $F = \oplus_{\alpha \in G} F_\alpha$ over k with 1 is called a *G-graded Banach algebra* if

(1) $(F_\alpha, \|\cdot\|)$ is a complete normed space for every $\alpha \in G$,

(2) $\|ab\| \leqq \|a\| \cdot \|b\|$ for any $\alpha, \beta \in G$ and $a \in F_\alpha$, $b \in F_\beta$.

Let M be an $I \times I$-matrix over F. We define a norm $\|M\|$ of M by

$$\|M\| = (\textstyle\sum_{i,j \in I} \|M^i_j\|^2)^{1/2}.$$

Then the algebra of $I \times I$-matrices over F becomes a Banach algebra. So we can define the exponential mapping exp by

$$\exp M = \textstyle\sum_{n=0}^{\infty} M^n / n!$$

for an $I \times I$-matrix M.

Theorem 3. *For an $I \times I$-matrix M over F we have*

$$\operatorname{sdet}(\exp M) = \exp(\operatorname{str} M).$$

The group of all invertible $I \times I$-matrices over F is a topological group and is called the *general linear group with parameters in F* and is denoted by $GL(I, F)$. For a closed subgroup H of $GL(I, F)$ we define

$$\mathcal{L}(H) = \{M \in gl(I, F) \mid \exp tM \in H \text{ for all } t \in k\}.$$

Then $\mathcal{L}(H)$ is a Lie subalgebra of $gl(I, F)$ and is called the *Lie algebra of H*.

Theorem 4. *Let $\mathcal{L}$ be a Lie algebra of a closed subgroup H of $GL(I, F)$. Assume that F_α is finite-dimensional over k for every $\alpha \in G$. Then $\{\exp X_1 \cdots \exp X_r \mid r \geqq 1, X_i \in \mathcal{L}\}$ is the connected component of H containing the identity element of H.*

Now we give two types of closed subgroups of $GL(I, F)$. The *special linear group* $SL(I, F)$ is the group of those matrices whose super determinants are 1. Next, let Ψ be an invertible $(-I) \times I$-matrix over F. $L_\Psi(I, F)$ denotes the group of *Ψ-preserving* matrices in $GL(I, F)$, that is,

$$L_\Psi(I, F) = \{M \in GL(I, F) \mid M^T \Psi M = \Psi\}.$$

Then the Lie algebras of $SL(I, F)$ and $L_\Psi(I, F)$ are $sl(I, F) = \{M \in gl(I, F) \mid \operatorname{str} M = 0\}$ and $\mathcal{L}_\Psi(I, F) = \{M \in gl(I, F) \mid M^T \Psi + \Psi M = 0\}$ respectively.

Let $L = \oplus_{\alpha \in G} L_\alpha$ be a G-graded algebra over k with operation $\langle\ ,\ \rangle$ satisfying (1) $\langle A, B\rangle = -\sigma(\alpha, \beta)\langle B, A\rangle$ and (2) $\sigma(\gamma, \alpha)\langle A, \langle B, C\rangle\rangle +$ cyclic $= 0$ for any $\alpha, \beta, \gamma \in G$ and $A \in L_\alpha$, $B \in L_\beta$, $C \in L_\gamma$. We call L a *Lie σ-algebra* following [7]. Suppose that L is finite-dimensional over k. By Ado's theorem ([7, Theorem 3]) L can be regarded as a subalgebra of the (G-graded) general linear Lie σ-algebra. Let $\mathcal{L} = \oplus_{\alpha \in G} L_\alpha \otimes_k F_{-\alpha}$. Then $\mathcal{L}$ is a subalgebra of our $gl(I, F)$ for a suitable G-set I. Then $H = \{\exp X_1 \cdots \exp X_r \mid r \geqq 1, X_i \in \mathcal{L}\}$ is a subgroup of $GL(I, F)$. Though H is not necessarily a closed subgroup, it may be permitted to say that H is the Lie group with parameters in F associated with L.

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