57. *Riemann-Hilbert-Birkhoff Problem for Integrable Connections with Irregular Singular Points*

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Let M be a complex manifold and let H be a divisor on M. Denote by $\Omega^p(*H)$ the sheaf over M of germs of meromorphic p-forms which are holomorphic in $M-H$ and have poles on H for $p=0, \cdots, n$. In case $p=0$, we use frequently $\mathcal{O}(*H)$ instead of $\Omega^0(*H)$.

We suppose throughout this paper that the divisor H has at most normal crossings.

Let $\mathcal{S}$ be a locally free sheaf of $\mathcal{O}(*H)$-modules of rank m and let ∇ be an integrable connection on $\mathcal{S}$. For any point $p \in H$, there exists an open set U in M containing p and a free basis $e_U=(e_{1U}, \cdots, e_{mU})$ of $\mathcal{S}$ over U. With respect to the free basis e_U, the connection ∇ is represented by $(d+\Omega_{eU})$, i.e.

$$\nabla(\langle e_{1U}, \cdots, e_{mU}\rangle u)=\langle e_{1U}, \cdots, e_{mU}\rangle(du+\Omega_{eU}u),$$

where Ω_{eU} is an m-by-m matrix of meromorphic 1-forms with poles at most on H and u is any m-vector of functions in $\mathcal{O}(*H)(U)$. If $f_U=\langle f_{1U}, \cdots, f_{mU}\rangle$ is another free basis of $\mathcal{S}$ over U, then there exists an m-by-m invertible matrix G of functions in $\mathcal{O}(*H)(U)$ such that

$$\langle f_{1U}, \cdots, f_{mU}\rangle=\langle e_{1U}, \cdots, e_{mU}\rangle G,$$

$$\nabla(\langle f_{1U}, \cdots, f_{mU}\rangle u)=\langle f_{1U}, \cdots, f_{mU}\rangle(du+(G^{-1}\{\Omega_{eU}G+dG\})u).$$

Let $x_1, \cdots, x_m$ be holomorphic local coordinates at p on U with $U\cap H=\{x_1\cdots x_{n''}=0\}$, then Ω_{eU} is written of the form

$$\Omega_{eU}=\sum_{i=1}^{n''} x^{-p_i}x_i^{-1}A_i(x)dx_i+\sum_{i=n''+1}^{n} x^{-p_i}A_i(x)dx_i,$$

where $p_i=(p_{i1}, \cdots, p_{in''}, 0, \cdots, 0) \in N^n$ and $A_i(x)$ is an m-by-m matrix of holomorphic functions in U for $i=1, \cdots, n$, and Ω_{eU} satisfies, by the integrability condition, $d\Omega_{eU}+\Omega_{eU}\wedge\Omega_{eU}=0$.

Suppose that for any point p on H

(H♯) *there exists an open set U containing p with holomorphic coordinates $x_1, \cdots, x_n$ and a free basis $\langle e_{1U}, \cdots, e_{mU}\rangle$ of $\mathcal{S}$ such that Ω_{eU} is written of the above form satisfying*

(H♯1) *$p_i=0$ or, $p_i>0$ and $A_i(0)$ has m distinct eigenvalues for all $i=1, \cdots, n''$.*

Let M^- be the real blow-up along H of M with the natural projection $pr: M^-\to M$. Denote by $\mathcal{A}^-$ the sheaf over M^- of germs of functions strongly asymptotically developable and write $\mathcal{A}^-(*H)$ for $\mathcal{A}^-\otimes_{pr^*\mathcal{O}} pr^*\mathcal{O}(*H)$. Denote by $GL(m, \mathcal{A}^-)$ and $GL(m, \mathcal{A}^-(*H))$ the sheaf of germs of m-by-m matricial invertible functions of which entries belong to $\mathcal{A}^-$ and $\mathcal{A}^-(*H)$, respectively, and denote by $GL(m, \mathcal{A}^-)_{I_m}$ the sheaf of germs of m-by-m invertible matricial functions strongly asymptotically developable to the m-by-m unit matrix I_m. Evidently, $GL(m, \mathcal{A}^-)_{I_m}$ is a subsheaf $GL(m, \mathcal{A}^-(*H))$: we denote by j the natural inclusion. For the above notation, we refer to the preceding article [10].

Then, we can assert

Theorem 1. *If the assumption* (H$\sharp$) *is satisfied for any point p on H, then there exists a locally free sheaf $\mathcal{F}$ of $\mathcal{A}^-(*H)$-modules over M^- and a connection $\nabla_{\mathcal{F}}$ on $\mathcal{F}$ such that*

(i) *there exists an isomorphism $g: \mathcal{F} \to pr^*\mathcal{S} \otimes_{pr^*\mathcal{O}(*H)} \mathcal{A}^-(*H)$ such that $g^{-1} \cdot (\nabla \otimes id) \cdot g = \nabla_{\mathcal{F}}$,*

(ii) *for any point p on H, there exists an open set U containing p such that the isomorphism class $[\mathcal{F}|_{U^-}]$ of $\mathcal{F}$ restricted on U^- belongs to $j_* H^1(U^-, GL(m, \mathcal{A}^-)_{I_m})$, where $U^- = pr^{-1}(U)$ and j_* is the natural inclusion induced by j,*

(iii) *for any point p on H and for an open set U containing p with holomorphic coordinates $x_1, \cdots, x_n$, $U \cap H = \{x_1 \cdots x_{n''} = 0\}$, there exist an m-by-m diagonal matrix D of functions in $\mathcal{O}(*H)(U)$ and upper triangular matrices T_i, $i=1, \cdots, n''$ such that*

(iii.a) *D, T_i $(i=1, \cdots, n'')$ are commutative each other,*

(iii.b) *for any point p' in $pr^{-1}(p)$ there exists an open set V^- containing p' and a free basis $\langle e(V^-)_1, \cdots, e(V^-)_m \rangle$ such that*

$$\nabla_{\mathcal{F}}(\langle e(V^-)_1, \cdots, e(V^-)_m \rangle v) = (\langle e(V^-)_1, \cdots, e(V^-)_m \rangle)(dv + \{dD(x) + \textstyle\sum_{i=1}^{n''} T_i x_i^{-1} dx_i\} v),$$

where v is any m-vector of functions in $\mathcal{A}^-(*H)(V^-)$.

Conversely,

Theorem 2. *For any locally free sheaf $\mathcal{F}$ of $\mathcal{A}^-(*H)$-modules over M^- and an integrable connection $\nabla_{\mathcal{F}}$ on $\mathcal{F}$ satisfying* (ii) *and* (iii), *there exists a locally free sheaf $\mathcal{S}$ of $\mathcal{O}(*H)$-modules over M and an integrable connection ∇ on $\mathcal{S}$ satisfying* (i).

In order to prove Theorem 2, we use the following lemma.

Lemma 1. *For a locally free sheaf $\mathcal{F}$ of $\mathcal{A}^-(*H)$-modules over M^- and an integrable connection $\nabla_{\mathcal{F}}$ on $\mathcal{F}$ satisfying* (ii), *there exists a locally free sheaf $\mathcal{S}$ of $\mathcal{O}(*H)$-modules over M and an integrable connection ∇ satisfying* (i).

Remark 1. For a polydisk U, $j_* H^1(U^-, GL(m, \mathcal{A}^-)_{I_m})$ is the unit element in $H^1(U^-, GL(m, \mathcal{A}^-(*H)))$ (see Majima [9] or [10]). This is the key to the proof of Lemma 1.

For the above $\mathcal{F}$ and $\nabla_{\mathcal{F}}$, the kernel sheaf Ker $\nabla_{\mathcal{F}}$ is a locally constant sheaf $\mathcal{C}$ on M^- which is thought to be a locally constant sheaf on

$M-H$. For any $p \in H$ and for any $p' \in pr^{-1}(p)$, take an open set $V^-(p')$ as in (iii.b), then $\{V^-(p') : p' \in pr^{-1}(p),\ p \in H\}$ is an open covering of a neighborhood of $pr^{-1}(H)$. Then, by (iii),

$$c(V^-(p')) = \langle e(V^-(p'))_1, \cdots, e(V^-(p'))_m \rangle \, \mathrm{ESS}\,(x(p))$$

is a free basis for $\mathcal{C}$ over $V^-(p')$, where $x(p)$ is the holomorphic local coordinate system chosen at $p=pr(p')$ and $\mathrm{ESS}\,(x(p))$ is a fundamental matrix of solutions of the system of equations

$$du + \{dD(x(p)) + \textstyle\sum_{i=1}^{n''} T_i(p) x_i^{-1}(p) dx_i(p)\} u = 0,$$

say, $\mathrm{ESS}\,(x(p)) = \exp\,(D(x(p))) \prod_{i=1}^{n''} x_i(p)^{T_i(p)}$. For $p', q' \in pr^{-1}(H)$, denote by $C_{V^-(p')V^-(q')}$ the transition matrix for $\mathcal{C}$ relative to the bases $c(V^-(p'))$, $c(V^-(q'))$, i.e.

$$c(V^-(p')) C_{V^-(p')V^-(q')} = c(V^-(q')).$$

And so, the matrix function

$$G_{V^-(p')V^-(q')} = \mathrm{ESS}\,(x(p)) C_{V^-(p')V^-(q')}\, \mathrm{ESS}\,(x(q))^{-1}$$

is the transition function for $\mathcal{F}$ relative to $e(V^-(p'))$, $e(V^-(q'))$. Therefore $G_{V^-(p')V^-(q')}$ is *strongly asymptotically developable* in $pr(V^-(p') \cap V^-(q')) - H$. In particular, if $pr(p')=pr(q')$, $G_{V^-(p')V^-(q')}$ is *strongly asymptotically developable to* I_m. *Conversely, given a locally constant sheaf* $\mathcal{C}$ *over* $M-H$ *and the matricial function* $\mathrm{ESS}\,(x(p))$ *for any* $p \in H$ *satisfying the above properties, there exists a locally free sheaf* $\mathcal{F}$ *over* M^- *of* $\mathcal{A}^-(*H)$*-modules and an integrable connection* $\nabla_{\mathcal{F}}$ *on* $\mathcal{F}$ *such that* (iii.b) *is satisfied and such that the kernel sheaf* $\mathrm{Ker}\,\nabla_{\mathcal{F}}$ *coincides with the given locally constant sheaf* $\mathcal{C}$. And so, by Theorem 2, *there exists a locally free sheaf* $\mathcal{S}$ *over* M *of* $\mathcal{O}(*H)$*-modules and an integrable connection* ∇ *on* $\mathcal{S}$ *satisfying* (i) *for this* $(\mathcal{F}, \nabla_{\mathcal{F}})$ *constructed from* $\mathcal{C}$ *and* $\mathrm{ESS}\,(x(p))$ *for any* $p \in H$.

Moreover, if $\mathcal{F} = \mathcal{F}' \otimes_{\mathcal{A}^-} \mathcal{A}^-(*H)$ with a locally free sheaf $\mathcal{F}'$ of $\mathcal{A}^-$-modules and if M is a Stein manifold or a projective manifold, by using Oka-Cartan's Theorem or Kodaira's vanishing theorem, we can prove the following (cf. [14], [5], [13]).

Theorem 3. *There exists a divisor* H' *on* M *and an integrable connection* ∇ *on the sheaf* $\mathcal{O}(*(H+H'))^m$, i.e. *a completely integrable system of Pfaffian equations on* M *with irregular singular points on* $(H+H')$, *such that* (i) *is satisfied.*

This theorem is classically formulated and proven by G. D. Birkhoff [2], [3] and reformulated locally by Balser-Jurkat-Lutz [1], Sibuya [16], [17] and Malgrange [12] in one variable case. On Rieman-Hilbert problem in several variables case, we refer to Deligne [4] (cf. Katz [8]), Gérard [5] and Suzuki [13].

The detail will be published elsewhere (see Majima [11]).

Correction (Proc. Japan Acad., 59A, 4 (1983)).
p. 147, line 16: For $\bigcap_{j=n''+1}^{j}$ read $\bigcap_{j=1}^{n''}$.

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