56. The μ-Number Constant Stratum of a Quasihomogeneous Function of Corank Two is Smooth

By Masahiko SUZUKI
Department of Mathematics, Faculty of Science, Chiba University

(Communicated by Kôsaku YOSIDA, M. J. A., May 12, 1983)

§ 0. **Introduction.** Let $f:(C^n, 0)\to(C, 0)$ be a germ of a quasihomogeneous function of a Milnor number μ with an isolated critical point. Let $\mathfrak{H}(n, 1)$ be a space of germs of holomorphic functions with an isolated critical point preserving the origin, namely

$$\mathfrak{H}(n, 1)=\{f \mid f:(C^n, 0)\to(C, 0) \text{ has an isolated critical point}\}.$$

V. I. Arnol'd conjectured in [1] that the μ-number constant stratum of a mini-transversal family of f is smooth, where a mini-transversal family implies a family which is transversal to the orbit of the action of the group of germs of biholomorphic mappings preserving the origin in $\mathfrak{H}(n, 1)$. He showed that this conjecture is affirmative for quasihomogeneous functions with inner modality $=0, 1$ in [1]. We showed it for them with inner modality $=2$ in [4]. Gabrielov and Kushnirenko showed it for all homogeneous functions with an isolated critical point in [2]. In this paper, we shall announce the affirmative answer to it for all quasihomogeneous functions of corank two with an isolated critical point.

Theorem. *Let f be the germ of corank two as above. Then the μ-number constant stratum of a mini-transversal family of f is smooth and its dimension is the number of generators of a monomial basis of a finite dimensional vector space $Q_f=m^2/m(\partial f/\partial x_1, \cdots, \partial f/\partial x_n)$ above and on the Newton boundary of f, where m is the maximal ideal of the local ring $C\{x_1, \cdots, x_n\}$.*

§ 1. **Sketch of a proof.** Let $f:(C^n, 0)\to(C, 0)$ be as above. We define $F:(C^n\times C^{\mu-1}, 0)\to(C, 0)$ to be the germ of the function

$$F(x, t)=f(x)+\sum_{i=1}^{\mu-1} t_i\phi_i(x),$$

where $\{\phi_1, \cdots, \phi_{\mu-1}\}$ is a monomial basis of Q_f. It is well known that the family F_t is a mini-transversal family of f in $\mathfrak{H}(n, 1)$. We denote by $\mathcal{S}_F$ the μ-number constant stratum of the family F_t, namely $\mathcal{S}_F=\{t\in C^{\mu-1} \mid \mu(F_t)=\mu(f)\}$ as a germ at the origin. If f is quasihomogeneous of type $(r_1, \cdots, r_n)$ i.e. $f(t^{r_1}x_1, \cdots, t^{r_n}x_n)=tf(x_1, \cdots, x_n)$ for any $t\in C$, then we see by Arnol'd [1] that $\mathcal{S}_F\supset\mathcal{A}_F$, where $\mathcal{A}_F$ is the germ at the origin of the set

$\{(t_1, \cdots, t_{\mu-1}) \in C^{\mu-1} \mid t_i=0$ for i for which quasidegree $(\phi_i)<1$ for $(r_1, \cdots, r_n)\}$.

We shall consider the following quasihomogeneous functions of three types

$$f_1(x, y)=x^a+y^b+g(x, y) \qquad a, b\geqq 3$$
$$f_2(x, y)=xf_1(x, y) \qquad a, b\geqq 2$$
$$f_3(x, y)=xyf_1(x, y) \qquad a, b\geqq 1,$$

where g is quasihomogeneous of type $(1/a, 1/b)$ and g does not contain the monomials x^a, y^b with non-zero coefficients and f_i has an isolated critical point. Then we can make the mini-transversal family F_{it} of f_i $(i=1, 2, 3)$ as follows;

$$F_1(x, y, t)=f_1(x, y)+\sum_{i=1}^{\mu_1-1} t_i\phi_{1i}(x, y),$$

where $\{\phi_{11}, \cdots, \phi_{1\mu-1}\}$ is a monomial basis of Q_{f_1} which does not contain the monomials x^iy^{b-1} $(i\geqq 1)$ and μ_1 is the μ-number of f_1.

$$F_2(x, y, t)=f_2(x, y)+\sum_{i=1}^{\mu_2-1} t_i\phi_{2i}(x, y)$$
$$F_2'(x, y, t)=f_2(x, y)+\sum_{i=1}^{\mu_2-1} t_i\phi_{2i}'(x, y),$$

where $\{\phi_{21}, \cdots, \phi_{2\mu_2-1}\}$ (resp. $\{\phi_{21}', \cdots, \phi_{2\mu_2-1}'\}$) is a monomial basis of Q_{f_2} which does not contain the monomials y^{b+i} $(i\geqq 1)$, $x^{i+1}y^{b-1}$ $(i\geqq 1)$ (resp. x^ay^i $(i\geqq 1)$) and μ_2 is the μ-number of f_2.

$$F_3(x, y, t)=f_3(x, y)+\sum_{i=1}^{\mu_3-1} t_i\phi_{3i}(x, y),$$

where $\{\phi_{31}, \cdots, \phi_{3\mu_3-1}\}$ is a monomial basis of Q_{f_3} which does not contain the monomials $x^{i+1}y^b$ $(i\geqq 1)$, y^{b+1+i} $(i\geqq 1)$ and μ_3 is the μ-number of f_3. Then we have the following three lemmata.

Lemma 1. *For the family F_{1t}, we have*

$$\mathcal{S}_{F_1}=\mathcal{A}_{F_1}.$$

Lemma 2. *If $b\leqq a$, then for the family F_{2t}, we have*

$$\mathcal{S}_{F_2}=\mathcal{A}_{F_2}.$$

If $a\leqq b$, then for the family F_{2t}', we have

$$\mathcal{S}_{F_2'}=\mathcal{A}_{F_2'}.$$

These lemmata are proved by using Brauner's theorem on topological types of irreducible plane curves and Zariski-Hironaka's theorem on topology of reducible plane curves etc. Proofs of the lemmata will appear in Topology (see [5]).

Now we shall prove the theorem as follows. Any germ f of corank two is equivalent to one of the germs of the functions

$$f_i(x, y)+z_1^2+\cdots+z_{n-1}^2 \qquad i=1, 2, 3.$$

It is based on Saito's theorem in [3]. We put

$$G_i(x, y, z, t)=F_i(x, y, z, t)+z_1^2+\cdots+z_{n-2}^2 \qquad i=1, 3$$
$$G_2(x, y, z, t)=\begin{cases}F_2(x, y, z, t)+z_1^2+\cdots+z_{n-2}^2\\ F_2'(x, y, z, t)+z_1^2+\cdots+z_{n-2}^2.\end{cases}$$

Note that $\mu(G_{tt})=\mu(f)$ if and only if $\mu(F_{tt})=\mu(f_t)$. Hence we have $\mathcal{S}_{G_t}=\mathcal{A}_{G_t}$ from the preceding lemmata. It is well known that for any mini-transversal family F_t of f in $\mathfrak{H}(n, 1)$, there exists a biholomorphic mapping $\tau:(C^{\mu-1}, 0)\to(C^{\mu-1}, 0)$ under which the germ F is equivalent to the germ G_t. By the mapping τ, we have an analytic isomorphism $\mathcal{S}_F\cong\mathcal{S}_{G_t}$. Hence the stratum $\mathcal{S}_F$ is smooth and its dimension is equal to the number of generators of a monomial basis of Q_f above and on the Newton boundary of f (see the definition of $\mathcal{A}_F$). This completes the proof of the theorem.

References

[1] V. I. Arnol'd: Normal forms of functions in the neighborhoods of degenerate critical points. Russian Math. Surveys, **29**, 10–50 (1974).

[2] A. M. Gabrielov and A. G. Kushnirenko: Description of the deformations with constant Milnor number for homogeneous functions. Funkt. Anal. i Ego Pril., vol. 9, no. 4, pp. 67–68 (1975).

[3] K. Saito: Quasihomogene isolierte Singularitäten von Hyperflächen. Invent. math., **14**, 123–142 (1971).

[4] E. Yoshinaga and M. Suzuki: Normal forms of non-degenerate quasihomogeneous functions with inner modality ≦4. ibid., **55**, 185–206 (1979).

[5] M. Suzuki: The stratum with constant Milnor number of a minitransversal family of a quasihomogeneous function of corank two (to appear in Topology).