53. A Generalization of the Fenchel-Moreau Theorem

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1. Let F be a real valued convex function defined on a locally convex space. The Fenchel-Moreau theorem is that $F(x)=F^{**}(x)$ if and only if F is lower semi-continuous at x ([1]). Many authors considered to generalize this theorem when F is a convex operator defined on a topological linear spaces to Riesz spaces. For example, J. Zowe has proved $F(x)=F^{**}(x)$ if F is continuous at x and x is an interior point of the domain of F. We shall consider the theorem in the case where F is not necessary continuous, nor the interior of the domain is non-empty. In the following, let X and Y be two Hausdorff locally convex topological vector spaces and Y is assumed further a Dedekind complete Riesz space (order complete vector lattice).

To relate the order structure and the topological structure, we demand furthermore that the linear topology of Y is *normal* i.e. the family of the following sets

$$(V+Y^+)\cap(V-Y^+)\,;\ V \text{ is an open set containing } 0,$$

constitutes a base of neighbourhoods of 0 for Y, where Y^+ denotes the totality of elements of Y equal to or greater than 0. A convex operator F defined on X into Y is to mean that the domain of F (denoted by $D(F)$) is a non-empty convex subset of X and

$$F(\alpha x_1+\beta x_2)\leqq \alpha F(x_1)+\beta F(x_2)$$

for $\alpha+\beta=1$ ($\alpha, \beta \in [0,1]$) and $x_1, x_2 \in D(F)$.

We shall define the conjugate function F^* of F. Let $L(X, Y)$ be the totality of all continuous linear operator from X to Y. For $A \in L(X, Y)$, we define F^* as follows:

$$F^*(A)=\sup\{A(x)-F(x)\,;\ x\in D(F)\}.$$

It is easy to see that F^* is a convex operator from $L(X, Y)$ to Y. Similarly, considering $X\subset L(L(X, Y), Y)$, we can define the double conjugate of F:

$$F^{**}(x)=\sup\{A(x)-F^*(A)\,;\ A\in L(X, Y)\}.$$

As usual, we define the subdifferential of F at $x\in D(F)$ with

$$\partial F(x)=\{A\in L(X, Y)\,;\ A(x)-A(x')\geq F(x)-F(x'),\ x'\in D(F)\}.$$

Furthermore, we shall define the y-subdifferential for $y \in Y^+$ as follows:

$$\partial_y F(x)=\{A\in L(X, Y)\,;\ A(x)-A(x')\geq F(x)-F(x')-y,\ x'\in D(F)\}.$$

It is easy to see that

(a) $\partial F(x)\neq\phi$ implies $\partial_y F(x)\neq\phi$,

(b) $0\leq y_1\leq y_2$ implies $\partial_{y_1}F(x)\subset\partial_{y_2}F(x)$.

In this note, we shall use the subdifferential $\partial_y F(x)$ mainly, although $\partial F(x)$ makes important roles in many papers.

2. We shall show the following lemmas.

Lemma 1. *$F^{**}(z)=F(z)$ iff* $\inf\{y>0,\ \partial_y F(z)\neq\phi\}=0$.

Proof. Since $\partial_y F(z)\ni A_1$, iff $A_1(z)-F(z)\geq A_1(x')-F(x')-y$ for $x'\in D(F)$ by definition, we have

$$\begin{aligned}F^{**}(z)&=\sup_{A\in L(X,Y)}(A(z)-F^*(A))\\&=\sup_A\{A(z)-\sup(A(x')-F(x'),\ x'\in D(F))\}\\&\geqq A_1(z)-\{A_1(z)-F(z)+y\}=F(z)-y.\end{aligned}$$

Hence $F^{**}(z)\geq F(z)$. Since $F^{**}(z)\leqq F(z)$ is always true, we have $F(z)=F^{**}(z)$.

Conversely, let $A\in D(F^*)$ and $y=F(z)-A(z)+F^*(A)$. We see easily $y\geq 0$ and $A\in\partial_y F(z)$.

Since $0\leqq\inf\{y\,;\ \partial_y F(z)\neq\phi\}\leqq\inf\{F(z)-A(z)-F^*(A)\,;\ A\in D(F^*)\}=\inf\{F^{**}(z)-A(z)+F^*(A)\,;\ A\in D(F^*)\}=0$, we have proved the lemma.

Lemma 2. *Let F be a convex operator defined on X to Y with $F(0)=0$ and continuous at 0. Then, every linear operator A on X to Y is continuous if $F(x)\geqq A(x)$ for all $x\in X$.*

Proof. For each symmetric open set V containing 0 of Y, there exists a neighbourhood U of 0 in X such that $F(U)\subset V$. Since $A(h)\leqq F(h)$ and $A(-h)\leqq F(-h)$, we have

$$A(h)\in(V+Y^+)\cap(V-Y^+)\qquad\text{for } h\in U.$$

Since the topology of Y is normal, A is continuous.

Lemma 3. *Let f be a positively homogeneous convex operator such that $D(f)$ is a convex cone of X, and let g be a positively homogeneous concave operator* ($-g$ is a convex operator) *with $D(g)=X$, and let $f(x)\geqq g(x)$ for $x\in D(f)$. Then*

$$h(x)=\inf\{f(y)-g(y-x)\ \text{for}\ y\in D(f)\}$$

is a positively homogeneous convex operator from X to Y.

By using the same method used in the proof of Hahn-Banach theorem, we can prove the following lemma.

Lemma 4. *For the positively homogeneous convex operator h defined in Lemma 3, there exists a linear operator ψ from X to Y such that*

$$\psi(x)\leqq h(x)\qquad\text{for } x\in X.$$

Hence, we have $g(x)\leqq\psi(x)$ for $x\in X$ and $\psi(x)\leqq f(x)$ for $x\in D(f)$.

3. We shall show now a generalization of the Fenchel-Moreau theorem.

Theorem 1. *Let F be a convex operator from X to Y such that $D(F)$ is not necessary to have an interior point and $S_z=\{y\in Y^+\,;\ F(U\cap D(F))\subset F(z)-y+Y^+$ for some neighbourhood U of $z\}\neq\phi$. Then*

(1) $\inf S_z = 0$ *implies* $F^{**}(z) = F(z)$.

Conversely,

(2) *If* $(Y^+)^\circ \neq \phi$, *then*

$$F(z) = F^{**}(z) + \inf S_z.$$

Hence $F^{**}(z) = F(z)$ *implies* $\inf S_z = 0$.

Proof. (1) For every $y \in S_z$, there exists a convex open set U of z (symmetric w. r. to z) such that

$$F(U \cap D(F)) \subset F(z) - y + Y^+.$$

Hence, we can easily find that

$$f(x) = F'_y(z, x) = \inf_{\lambda > 0} \frac{1}{\lambda}\{F(z + \lambda x) - F(z) + y\} \geqq -y \qquad \text{for } x \in U - z.$$

If we define a gauge function $G(x) = \inf\{\lambda > 0,\ x \in \lambda(U - z)\}$, then $G(x)$ is a continuous convex function on X, so that $g(x) = -2G(x)y$ is a concave continuous operator from X to Y and $f(x) \geqq g(x)$ for all $x \in D(f)$.

By Lemma 4, there exists a linear operator ψ such that $f(x) \geqq \psi(x) \geqq g(x)$. But by Lemma 2, ψ is continuous since $g(x)$ is continuous. Hence

$$0 \leqq \inf\{y\,;\ \partial_y F(z) \neq \phi\} \leqq \inf S_z = 0.$$

By Lemma 1, we find $F^{**}(z) = F(z)$.

(2) It is easy to see that Y^+ is closed if $(Y^+)^\circ \neq \phi$. Let $y_0 = \inf S_z > 0$ and y is not greater than y_0, then there exists some positive number $\varepsilon > 0$ with $(1+\varepsilon)y$ is not an element of S_z. Hence, there exists a sequence $\{z_\lambda\}$ convergent to z where $F(z_\lambda)$ is not greater than $F(z) - (1+\varepsilon)y$. From this fact, we find that

(*) $F'_y(z, z_\lambda - z)$ is not greater than $-\varepsilon y$.

Suppose $y \in (Y^+)^\circ$. Then we shall prove that $\partial_y F(z) = \phi$. If $\psi \in \partial_y F(z)$, then it follows

$$\psi(x) \leqq f(x) \qquad \text{for } x \in D(f)$$

and so by (*) $\psi(z_\lambda - z)$ is not greater than $-\varepsilon y$.

But, there exists a neighbourhood U of 0 such that $x \geqq -\varepsilon y$ for all $x \in U$ and $\psi(z_\lambda - z) \notin U$. Hence, ψ is not continuous and so it is impossible.

In general case, suppose $y \in Y^+$, then there exists $y_1 \geq y$ such that $y_1 \in (Y^+)^\circ$ and y_1 is not greater than y_0. Since $\partial_{y_1} F(z) = \phi$, we have $\partial_y F(z) = \phi$, as remarked in (b) of § 1 of this note. Hence, we have

$$F^{**}(z) = F(z) \quad \text{implies} \quad \inf S_z = 0.$$

By the same argument, we have

$$F(z) = F^{**}(z) + \inf S_z.$$

Remark 1. If we don't assume $(Y^+)^\circ \neq \phi$, then (2) of Theorem 1 is not true in general, although (1) is true in any case.

Remark 2. There exists some example that the convex operator

F is not continuous in everywhere and Theorem 1 is still valid. For the case $Y=l_p(\infty>p\geqq 1)$, we know that $(Y^+)^\circ=\phi$. In this case we have the following theorem.

Theorem 2. *Let $F=(f_1, f_2, \cdots)$ be a convex operator from X to l_p. Suppose that $f^{**}(z)$ is defined for $z\in D(F)$, then $F^{**}(z)=F(z)$ iff each f_i $(i=1,2,\cdots)$ is lower semi-continuous at z.*

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