6. Global Solutions of the Nonlinear Schrödinger Equation in Exterior Domains

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(Communicated by Kôsaku YOSIDA, M. J. A., Jan. 12, 1983)

§ 1. **Introduction and theorem.** We consider the following initial boundary value problem for the nonlinear Schrödinger equation in an exterior domain $\Omega \subset \boldsymbol{R}^n$, $n \geqq 3$:

$$i\frac{\partial u}{\partial t} = \Delta u + \lambda |u|^p u \qquad \text{in } [0, \infty) \times \Omega, \tag{1.1}$$

$$u(0, x) = u_0(x), \tag{1.2}$$

$$u|_{\partial\Omega} = 0. \tag{1.3}$$

Here λ is a real constant and p is an even integer with $p \geqq 2$. The domain Ω is the exterior of a compact set in $\boldsymbol{R}^n$, $n \geqq 3$, with the smooth boundary $\partial\Omega$. In the present paper we shall prove that Problem (1.1)–(1.3) has a unique global solution for small initial data under a certain assumption on the shape of Ω, which indicates that Ω is "non-trapping" in the sense of Vainberg [2] and Rauch [3].

For the Cauchy problem, namely the case of $\Omega = \boldsymbol{R}^n$, the above problem has been extensively studied. For the exterior problem, however, we know only the work of Brézis and Gallouet [1]. In [1] they treated Problem (1.1)–(1.3) only for the case of $n=2$.

We shall first give some notations. For an open set D in $\boldsymbol{R}^n$, let $H^m(D)$, $H_0^m(D)$, $L^2(D)$, $L^1(D)$ and $C_0^\infty(D)$ denote the standard function spaces. We shall fix $R>0$ such that $\partial\Omega \subset \{x \in \boldsymbol{R}^n ; |x| < R\}$. For any $r \geqq R$, we denote the set $\{x \in \Omega ; |x| < r\}$ by Ω_r. We shall often abbreviate $\left(\frac{\partial}{\partial x}\right)^\alpha$ and $\left(\frac{\partial}{\partial t}\right)^j$ to ∂_x^α and ∂_t^j respectively, where α is a multi-index and j is a nonnegative integer. For $a \in \boldsymbol{R}^1$ we denote by $[a]$ the greatest integer that is not larger than a.

Let $G = G(t, x, x_0)$ be the Green function for the following problem:

$$(\partial^2/\partial t^2 - \Delta_x)G = 0 \qquad \text{in } (0, \infty) \times \Omega,$$

$$\lim_{t \to +0} \frac{\partial^j G}{\partial t^j} = \begin{cases} 0, & j=0, \\ \delta(x - x_0), & j=1, \end{cases}$$

$$G|_{x \in \partial\Omega} = 0,$$

where x_0 is an arbitrary point of Ω. For any $\psi(x) \in C_0^\infty(\boldsymbol{R}^n)$ we define $f(t, x, x_0)$ by

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$$f(t, x, x_0)=\psi(x)G(t, x, x_0).$$

For any $v \in L^2(\Omega)$ with its support included in Ω_R we define $(Fv)(t, x)$ by

$$(Fv)(t, x)=\int_\Omega f(t, x, x_0)v(x_0)dx_0.$$

Now we shall make the following assumption on the domain Ω:

Assumption [A]. For each $s>0$ and each nonnegative integer N, there exist two positive constants T_0 and C such that

$$\sup_{t\in[T_0,T_0+s]} \sum_{|\alpha|+j\leqq N} \|\partial_x^\alpha \partial_t^j (Fv)(t, \cdot)\|_{L^2(\Omega)} \leqq C\|v\|_{L^2(\Omega)}$$

for any $v \in L^2(\Omega)$ with its support included in Ω_R. T_0 and C depend on N, R, Ω and a function $\psi(x)$.

Remark 1.1. Assumption [A] is almost the same as assumptions that Vainberg and Rauch assumed in their works (see Vainberg [2, the hypothesis D', p. 11] and Rauch [3, the hypothesis (9.3), p. 476]). Assumption [A] implies that singularities of the Green function of a wave equation in the exterior domain Ω go to infinity as $t\to\infty$. For example it is known that if the complement of Ω is convex, Assumption [A] is satisfied (see Melrose [4] and Rauch [3]).

Our main theorem is the following:

Theorem 1.1. *Let $n\geqq 3$ and p be an even integer with $p\geqq 2$. We assume that Assumption* [A] *is satisfied. Then, for each integer $N\geqq 3([n/2]+3)/2$, there exists a positive constant ε such that if the initial data $u_0(x)$ satisfy*

$$\sum_{|\alpha|\leqq 2N} \|\partial_x^\alpha u_0\|_{L^2(\Omega)} + \sum_{|\alpha|\leqq 2N-[n/2]-2} \|\partial_x^\alpha u_0\|_{L^1(\Omega)} < \varepsilon \tag{1.4}$$

and the compatibility condition, Problem (1.1)–(1.3) *has a unique global solution:*

$$u(t, \cdot) \in \left\{\bigcap_{k=0}^{N-1} C^k([0, \infty); H_0^1(\Omega)\cap H^{2(N-k)}(\Omega))\right\} \cap C^N([0, \infty); L^2(\Omega)).$$

Remark 1.2. (i) In the statement of Theorem 1.1 the compatibility condition means that the boundary values of $\partial_t^j u|_{t=0}$ $(0\leqq j\leqq N-1)$ are compatible with the boundary condition. For details, see Mizohata [5].

(ii) From the Sobolev imbedding theorem, it follows that a solution given by Theorem 1.1 is a classical solution.

In the present paper only the sketch of the proof of Theorem 1.1 will be described below. Details will be published elsewhere.

§ 2. Proof of Theorem 1.1.

For the local existence and uniqueness of a solution to Problem (1.1)–(1.3) we have the following theorem:

Theorem 2.1. *Let N be an integer with $N\geqq[n/2]+1$ and let p be an even integer with $p\geqq 2$. Then there exist two positive constants ε' and T such that if the initial data $u_0(x)$ satisfy*

$$(2.1)\qquad \sum_{|\alpha|\leqq 2N} \|\partial_x^\alpha u_0\|_{L^2(\Omega)} < \varepsilon'$$

and the compatibility condition, Problem (1.1)–(1.3) *has a unique local solution*:

$$u(t,\cdot)\in\left\{\bigcap_{k=0}^{N-1} C^k([0,T];H_0^1(\Omega)\cap H^{2(N-k)}(\Omega))\right\}\cap C^N([0,T];L^2(\Omega)).$$

We can prove this theorem by choosing a proper function space and applying the contraction mapping principle in it.

In addition to Theorem 2.1 we need an a priori estimate to obtain Theorem 1.1. The proof of establishing the a priori estimate is based on a decay estimate and an energy estimate for the linear problem. Let us consider the inhomogeneous linear Schrödinger equation:

$$(2.2)\qquad i\frac{\partial u}{\partial t}=\Delta u+f \qquad \text{in } [0,\infty)\times\Omega,$$

$$(2.3)\qquad u(0,x)=u_0(x),$$

$$(2.4)\qquad u|_{\partial\Omega}=0.$$

In what follows we assume that f is a smooth function such that all norms of f which will appear in the following lemmas are bounded. For $1\leqq q\leqq\infty$, $k\in \boldsymbol{R}^1$ and a nonnegative integer L, we define $[v;q,k,L](t)$ by

$$[v;q,k,L](t)=\sup_{s\in[0,t]}\sum_{|\alpha|+2j\leqq L}(1+s)^k\|\partial_x^\alpha\partial_s^j v(s,\cdot)\|_{L^q(\Omega)}.$$

For a solution $u(t,x)$ of Problem (2.2)–(2.4) we have the following two lemmas on the decay estimate and the energy estimate:

Lemma 2.1. *Let Assumption* [A] *be satisfied and* $n\geqq 3$. *Then, for each nonnegative integer* L, *the solution* $u(t,x)$ *of Problem* (2.2)–(2.4) *satisfies the following decay estimate*:

$$\begin{aligned}(2.5)\quad [u;\infty,n/2,2L](t)\leqq C_L\{&\sum_{|\alpha|\leqq 2L+[n/2]+4}\|\partial_x^\alpha u_0\|_{L^1(\Omega)}\\ &+\sum_{|\alpha|\leqq 2L+2[n/2]+5}\|\partial_x^\alpha u_0\|_{L^2(\Omega)}+[f;1,n/2,2L+[n/2]+4](t)\\ &+[f;2,n/2,2L+2[n/2]+5](t)\},\qquad t\geqq 0,\end{aligned}$$

where C_L *is a positive constant depending only on* n, L *and* Ω.

For a wave equation with the homogeneous Dirichlet boundary condition uniform decay estimates of the type (2.5) has been obtained recently by Shibata [7] and [8]. The strategy of the proof of Lemma 2.1 follows Shibata [8].

Lemma 2.2. *Let* $n\geqq 3$. *Then, for each nonnegative integer* L, *the solution* $u(t,x)$ *of Problem* (2.2)–(2.4) *satisfies the following energy estimate*:

$$(2.6)\quad [u;2,0,2L](t)\leqq \overline{C}_L\{\sum_{|\alpha|\leqq 2L}\|\partial_x^\alpha u_0\|_{L^2(\Omega)}+[f;2,n/2,2L](t)\},\quad t\geqq 0,$$

where $\overline{C}_L$ *is a positive constant depending only on* n, L *and* Ω.

By using Lemmas 2.1, 2.2 and the technique due to Matsumura and Nishida [6], we can establish the desired a priori estimate so that we can continue local solutions given by Theorem 2.1 to arbitrary times. Thus, we obtain Theorem 1.1. The original proof of Theorem 1.1, which was based on the Nash-Moser implicit function theorem, was simplified owing to Prof. Y. Shibata's advice that Matsumura and Nishida's method could be applied to our problem.

Finally the author would like to express his deep gratitude to Profs. T. Kato, M. Murata, M. Tanikawa and especially to Prof. Y. Shibata for their valuable advice.

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